Answers, hints and some solutions to exercises
in ODE and modeling MMG511/TMV162. Spring 2016.
Banach spaces. Lipschitz functions. Picard-Lindelöf theorem.
Gronwall inequality. Uniqueness of solutions.

1. Prove that the space $C(I)$ of continuous vector valued functions on a bounded closed interval $I = [a, b]$, $\varphi : I \longmapsto R^n$, with the norm $\|\varphi\|_{C(I)} \overset{def}{=} \sup_{t\in I} |\varphi(t)|$ is a Banach space, namely that $\|\varphi\|_{C(I)}$ satisfies axioms for a norm and that this space is complete with respect to it.

2. Find a bounded sequence $\{f_m\}_{m=1}^{\infty}$ in $C(I)$, $I = [a, b]$ such that there is no convergent subsequence.

Answer. $f_m(t) = \sin(tm)$. One can argue by contradiction. Suppose there is a uniform limit $f(t)$ for some subsequence $\{f_{m_l}(t)\}_{l=1}^{\infty}$ and show that it is impossible because on a short interval $|t - t_0| \leq \varepsilon$ where $|f(t) - f(t_0)| \leq \delta << 1$ functions $f_{m_l}(t)$ will attain ALL walues between 1 and -1 for m_l large enough (depending on ε, for example $m_l > 2\pi(2\varepsilon)$).

3. Let $K = K(x, y) : [a, b] \times [a, b] \longmapsto R$ be continuous with $0 \leq K(x, y) \leq d$ for all $x, y \in [a, b]$. Let $2(b - a)d \leq 1$

and $u_0(x) \equiv 0$, $v_0(x) \equiv 1$. Then both iterates

$$
\begin{aligned}
u_{n+1}(x) &= \int_a^b K(x, y)u_n(y)dy + 1 \\
v_{n+1}(x) &= \int_a^b K(x, y)v_n(y)dy + 1
\end{aligned}
$$

converge to a unique solution to the equation

$$
u(x) = \int_a^b K(x, y)u(y)dy + 1, \; x \in [a, b]
$$

Hint. Exersises 3 and 4 are solved in two steps.

At the first step one finds a closed ball $\overline{B}(R, 0) =\{\phi(t) : \sup_{t\in I} |\phi(t)| \leq R\}$ in $C(I)$ such that the operator $K(u) = \int_a^b K(x, y)u(y)dy + 1$ in the Ex. 3, correspondingly operator $T(x) = \int_0^2 B(t, s)x(s)ds + g(t)$ in the Ex.4 maps this ball to itself. In particular in the case of Ex. 4 one finds R such that for all functions $\phi(t)$ with $\|\phi\| = \sup_{t\in I} |\phi(t)| \leq R$ where I is the interval of integration, it is valid $\|\mathcal{T}(u)\| = \sup_{t\in I} |\mathcal{T}(u)(t)| \leq R$.One uses the inequality for integrals: $\left|\int f(s)ds\right| \leq \int |f(s)|\, ds$ and the triangle inequality: $\|\phi + \lambda\| \leq \|\phi\| + \|\lambda\|$.

At the second step one shows that $K(u)$ and $T(u)$ are a contractions on the chosen $\overline{B}(R, 0)$, namely for Ex. 4 one estimates the norm

$$
\|\mathcal{T}(u) - \mathcal{T}(w)\| = \sup_{x\in I} \left| \int_a^b B(x, y)u(y)dy - \int_a^b B(x, y)w(y)dy \right|
$$

$\theta\,\|\mathcal{T}(u) - \mathcal{T}(w)\| = \theta\,[\sup_{x\in I} |u(x) - w(x)|]$ with a positive constant $0 < \theta < 1$ strictly smaller that 1. One can also carry out the second step first.

4. Consider the following operator

$$\mathcal{T}(x)(t) = \int_0^2 B(t,s)x(s)ds + g(t),$$

with $B(t,s)$ and $g(t)$ continuous functions and $|B(t,s)| < 0.25$ for all $t, s \in [0,2]$ acting in the Banach space $C([0,2])$ of continuous functions with norm $\|x\| = \sup_{t\in[0,2]} |x(t)|$.

Show using Banach's contraction principle that $T(x)(t)$ has a fixed point.

5. Consider the following (nonlinear!) operator

$$K(x)(t) = \int_0^2 B(t,s)\left[x(s)\right]^2 ds + g(t),$$

acting on the Banach space $C([0,2])$ of continuous functions with norm $\|x\| = \sup_{t\in[0,2]} |x(t)|$. Here $B(t,s)$ and $g(t)$ are continuous functions and $|B(t,s)| < 0.5$ for all $t, s \in [0,2]$. Estimate the norm $\|K(x) - K(y)\|$ for the operator $K(x)(t)$. Find requirements on the function $g(t)$ such that Banach's contraction principle implies that $K(x)(t)$ has a fixed point.

Hint. This exersise is solved similarly to exercises 3 and 4 with an important difference in the result. The non-linearity of the operator in this case makes that it is contraction only on a small ball around zero function and only for a function $g(t)$ that is small enough.

Solution.

Banach's contraction principle. Let B be a nonempty closed subset of a Banach space X and let the non-linear operator $K : B \to B$ be a contraction.

$$\|K(x) - K(y)\| \le \theta \|x - y\|, \theta < 1$$

Then K has a fixed point $\overline{x} = K(\overline{x})$ such that

$$\|K^n(x) - \overline{x}\| \le \frac{\theta^n}{1-\theta}$$

for any $x \in B$.

We like to have the estimate $\|K(x) - K(y)\| \le \theta \|x - y\|$ for x, y in some closed subset B of $C([0,2])$.

$\|K(x) - K(y)\| \le \sup_{t\in[0,2]} \left| \int_0^2 B(t,s) \left([x(s)]^2 - [y(s)]^2 \right) ds \right|$

$= \sup_{t\in[0,2]} \left| \int_0^2 B(t,s) \left(x(s) - y(s)\right)\left(x(s) + y(s)\right) ds \right| \le$

$\left| \int_0^2 \sup_{t,s\in[0,2]} B(t,s) ds \right| \|x - y\| \|x + y\| \le \|x - y\| \|x + y\| \le \|x - y\| \left(\|x\| + \|y\|\right)$

We can choose a ball $B \subset C([0,2])$ such that for any x, $y \in B$ it follows $\|x\| + \|y\| \le \theta < 1$, for example B can be taken as a set of functions with $\|x\| \le 1/4$. On this set K will be a contraction because $\|K(x) - K(y)\| \le \|x - y\| (0.5)$.

To apply Banachs principle we need also that K maps B into itself, namely that $\|K(x)\| \le 1/4$ for $\|x\| < 1/4$.

It gives a requirement on function $g(t)$. Estimate the operator $K:$
$\|K(x)\| \le \sup_{t\in[0,2]} \left|\int_0^2 B(t,s)\left([x(s)]^2\right) ds\right| + \sup_{t\in[0,2]} |g(t)| \le \|x\|^2 + \|g\|$
If $\|x\| < 1/4$ then we like to have that $\|K(x)\| \le 1/4$ that follows if $\|K(x)\| \le 1/16 + \|g\| \le 1/4$
It is satisfied if $\|g\| \le 3/16$. Therefore for $\|g\| \le 3/16$ the operator K has a unique fixed point in the ball $B : \|x\| \le 1/4$. vskip 0.3cm 6. Consider I.V.P. $y' = y^2 + t$; $y(1) = 0$. Reduce it to an integral equation and calculate successive approximations y_0, y_1, y_2. Find time interval for which successive approximations converge.

7. Show that if $f \in C^1(D)$ then it is locally Lipschitz in D.

8. Are following functions Lipschitz near zero?
(i) $f(x) = \frac{1}{1-x^2}$.
(ii) $f(x) = |x|^{1/2}$.
(iii) $f(x) = x^2 \sin\left(\frac{1}{x}\right)$.
Answer: (i) - Lipschitz, (ii) - not Lipschitz, (iii) Lipschitz after defining $f(0) = 0$.

9. Prove that I.V.P. for a linear ODE $x' = A(t)x$, $x(t_0) = x_0$, with $x(t)$, $x_0 \in R^n$, $A(t)$- $n \times n$ matrix, $A(t) \in C(R)$ has a unique solution for arbitrary (t_0, x_0).

Answer. Function $f(x,t) = A(t)x$ has continuous partial derivatives with respect to coordinates x_i of $x \in R^n$ that are components $A_{pm}(t)$ of A. It makes function $f(x,t) = A(t)x$ Lipschitz with respect to x and implies the uniquness of solutions. The Lipschitz constant can be chosen as $L = \sup_{t\in[t_0,T],p,m} (|A_{pm}(t)|) < \infty$ because $A(t)$ is continuous on $[t_0, T] \subset R$.

10. Prove the particular case of the Gronwall inequality:

$$y(t) \le \lambda \exp\left(\int_a^t \mu(s)ds\right),$$

in the case λ is a constant , $\mu(t) > 0$ and $y(t)$ has for $t \in [a,b]$ the property

$$y(t) \le \lambda + \left(\int_a^t \mu(s)y(s)ds\right),$$

Hint. Integrate the general Gronwall inequality with constant λ by parts using that $\frac{d}{dt}\left(\int_a^t \mu(s)ds\right) = \mu(t)$.

11. Find general solution to following ODE: $x' = x(1-x) - c$. Investigate the behaviour of solutions depending on initial data $x(0) > 0$ and on the constant $c > 0$.

Observe that the equation describes the evolution of a population x with limited growth and harvest rate c. Can one find an optimal harvest?

Hint. Observe that the equation is with separablr variables and that the analytical form of the solution depends on how many real roots has the equation $x(1-x) - c = 0$. These roots are dependent on turn on the constant c.

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