24. For each of the following systems show that the indicated region R is positively invariant:

(a) $\dot{x}_1 = 2x_1x_2, \quad \dot{x}_2 = x_2^2, \quad R = \{(x_1,x_2)|x_2 \geqslant 0\};$

(b) $\dot{x}_1 = -\alpha x_1 + x_2, \quad \dot{x}_2 = (\beta - \alpha)x_2; \quad \alpha, \beta$ constant, $R = \{(x_1,x_2)|x_2 = \beta x_1\};$

(c) $\dot{x}_1 = -x_1 + x_2 + x_1(x_1^2 + x_2^2), \quad \dot{x}_2 = -x_1 - x_2 + x_2(x_1^2 + x_2^2),$ $R = \{(x_1,x_2)|x_1^2 + x_2^2 < 1\};$

(d) $\dot{x}_1 = x_1(x_2^2 - x_1), \dot{x}_2 = -x_2(x_2^2 - x_1), \quad R = \{(x_1,x_2)|x_1 > x_2^2\}.$

26. Show that the polar form of the non-linear system

$$\dot{x}_1 = -x_2 + x_1(1 - x_1^2 - x_2^2), \qquad \dot{x}_2 = x_1 + x_2(1 - x_1^2 - x_2^2)$$

is given by

$$\dot{r} = r(1 - r^2), \qquad \dot{\theta} = 1.$$

Solve this equation subject to the initial conditions $r(0) = r_0, \theta(0) = \theta_0$ at $t = 0$ to obtain

$$r(t) = r_0/(r_0^2 + (1 - r_0^2)e^{-2t})^{1/2}.$$

Plot the graph of $r(t)$ against t for

(a) $0 < r_0 < 1$, (b) $r_0 = 1$, (c) $r_0 > 1$,

and obtain the phase portrait of the system. Can the phase portrait be sketched more easily from the polar differential equation?

Hints and answers

24. (a) Let $x_2(0) \geqslant 0$; then $\dot{x}_2 \geqslant 0$ implies $x_2(t) \geqslant x_2(0)$ for all positive t;

(b) For $x_2 = \beta x_1$, $\dot{x}_1 = (\beta - \alpha)x_1$ and $\dot{x}_2 = (\beta - \alpha)x_2$. Thus $\dot{x}_2 = \beta \dot{x}_1$;

(c) In polar coordinates $\dot{r} = -r(1 - r^2)$ and hence $\dot{r} < 0$ for $r < 1$;

(d) The trajectories lie on the family of hyperbolae $x_1x_2 = C$. Show that the boundary of the indicated region is a parabola of fixed points.