

### 1. Lyapunov function and stability of stationary points.

Consider the system of ODE: 
$$\begin{aligned}x' &= y \\y' &= -y + y^3 - x^5\end{aligned}$$

Write the definition of asymptotically stable stationary point.

Find Lyapunov function  $V(x, y)$  for the given equation and show that stationary point in the origin is asymptotically stable. (2p)

**Hint.** Use  $V(x, y)$  in the form  $V(x, y) = ax^6 + by^2$  and choose the parameters  $a, b$  so that  $V(x, y)$  would be a Lyapunov function.

**Solution** A stationary point  $x_*$  is asymptotically stable if there is a neighborhood  $N$  of this point such that for any initial point  $x(o) \in N$  the corresponding trajectory  $x(t) \rightarrow x_*$  for  $t \rightarrow +\infty$ .

$$V'(x, y) = \frac{\partial V}{\partial x}y + \frac{\partial V}{\partial y}(-y + y^3 - x^5) = 6ax^5y - 2by^2 + 2by^4 - 2bx^5y$$

Taking  $6a = 2b$  and  $a = 1$  makes  $V'(x, y) = 6y^2(1 - y^2) < 0$  for  $|y| < 1$  and  $y \neq 0$ .

For  $y = 0$   $y' = -x^5$  and therefore  $y' = 0$  only for  $x = 0$ . It implies that the origin is asymptotically stable.

### 1. Lyapunovs funktioner och stabilitet hos stationära punkter.

a) Formulera ett criterium för asymptotiskt stabil stationär punkt till en ODE med hjälp av en Lyapunovs funktion som inte är stark Lyapunovs funktion. (2p)

b) Betrakta ekvationssystemet: 
$$\begin{aligned}x' &= -y/3 - x(3x^2 + y^2) \\y' &= x - y(3x^2 + y^2)\end{aligned}$$

Hitta en stark Lyapunovs funktion  $V(x, y)$  för att visa att stationära punkten i origo är asymptotiskt stabil. (2p)

**Tips.** Använd  $V(x, y)$  på formen  $V(x, y) = ax^2 + y^2$  och välj parametern  $a$  så att  $V(x, y)$  blir en stark Lyapunovs funktion.

$V(x, y)$  kan väljas som  $V(x, y) = 3x^2 + y^2$ .

2. **Lyapunovs functions and stability of stationary points.** Formulate the criterion for asymptotic stability of a stationary point of an ODE using only a weak Lyapunov function.

Consider the system of equations: 
$$\begin{cases} x' = -x + y^2 \\ y' = -xy - x^2 \end{cases}$$

Show that  $V(x, y) = x^2 + y^2$  is a weak Lyapunov function and decide if the stationary point at the origin is asymptotically stable. (4p)

$$\frac{d}{dt}(V) = \nabla V \cdot \begin{bmatrix} -x + y^2 \\ -xy - x^2 \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix} \cdot \begin{bmatrix} -x + y^2 \\ -xy - x^2 \end{bmatrix} = 2x(-x + y^2) + 2y(-xy - x^2) = -2x^2 - 2x^2y = -2x^2(1 + y);$$

For  $|y| < 1$   $\frac{d}{dt}(V) \leq 0$ . It implies that the origin is a stable stationary point. On the line  $x = 0$   $\frac{d}{dt}(V) = 0$  so  $V$  is only a weak Lyapunovs funktion. But we observe that on the line  $x = 0$  the velocity is not zero  $-x + y^2 \neq 0$  except the stationary point itself. It implies that the origin is an asymptotically stable stationary point.

2. **Ljapunovs functions and stability of stationary points.**

Consider the system of equations: 
$$\begin{cases} x' = -x + 2xy^2 \\ y' = -(1 - x^2)y^3 \end{cases}$$

Show that the origin is an stable fixed point. (4p)

**Solution**

$$V(x, y) = x^2 + y^2$$

$$V' = 2x(-x + 2xy^2) + 2y(-(1 - x^2)y^3) = -2x^2 + 4x^2y^2 - 2y^4(1 - x^2) = -2x^2(1 - 2y^2) - 2y^4(1 - x^2)$$

We see that  $V' < 0$  for  $|x| < 1$  and  $|y| < \sqrt{1/2}$

2. **Ljapunovs functions and stability of fixed points.**

Consider the system of equations: 
$$\begin{cases} x' = -x + 2xy \sin(y) \\ y' = -\cos(x)y \end{cases}$$

Investigate stability of the fixed point in the origin. (4p)

We try if function  $V(x, y) = x^2 + y^2$  is a Lyapunovs funktion in some neighbourhood of origin.

$$V' = 2x(-x + 2xy \sin(y)) + 2y(-\cos(x)y) = -2x^2 - 2y^2 + 4x^2y \sin(y) + 2y^2(1 - \cos(y)) = -2(x^2 + y^2) + 2(x^4 + y^4) + x^2O(y^3) + 2y^2O(y^2)$$

where  $O(z)/z < const$  when  $z \rightarrow 0$ . It shows that in a small neighbourhood of the origin all terms in  $V'$  are dominated by  $-2(x^2 + y^2)$  and that  $V$  is a strong Ljapunovs funktion and the origin is an asymptotically stable point for the system.