Problems on stability by the second Liapunovs method

1. Show that $V(x_1, x_2) = x_1^2 + x_2^2$ is a strong Liapunov function at the origin for each of the following systems:

(a) $\dot{x}_1 = -x_2 - x_1^3, \quad \dot{x}_2 = x_1 - x_2^3;$

(b) $\dot{x}_1 = -x_1^3 + x_2 \sin x_1, \quad \dot{x}_2 = -x_2 - x_1^2 x_2 - x_1 \sin x_1;$

(c) $\dot{x}_1 = -x_1 - 2x_2^2, \quad \dot{x}_2 = 2x_1 x_2 - x_2^3;$

(d) $\dot{x}_1 = -x_1 \sin^2 x_1, \quad \dot{x}_2 = -x_2 - x_2^5;$

(e) $\dot{x}_1 = -(1 - x_2) x_1, \quad \dot{x}_2 = -(1 - x_1) x_2.$

3. Show that $V(x_1, x_2) = x_1^2 + x_2^2$ is a weak Liapunov function for the following systems at the origin:

(a) $\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 - x_2^3 (1 - x_1^2)^2;$

(b) $\dot{x}_1 = -x_1 + x_2^2, \quad \dot{x}_2 = -x_1 x_2 - x_1^2;$

(c) $\dot{x}_1 = -x_1^3, \quad \dot{x}_2 = -x_1^2 x_2;$

(d) $\dot{x}_1 = -x_1 + 2x_1 x_2^2, \quad \dot{x}_2 = -x_1^2 x_2^3.$

Which of these systems are asymptotically stable?

4. Prove that if V is a strong Liapunov function for $\dot{\mathbf{x}} = -\mathbf{X}(\mathbf{x})$, in a neighbourhood of the origin, then $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$ has an unstable fixed point at the origin. Use this result to show that the systems:

(a) $\dot{x}_1 = x_1^3, \quad \dot{x}_2 = x_2^3;$

(b) $\dot{x}_1 = \sin x_1, \quad \dot{x}_2 = \sin x_2;$

(c) $\dot{x}_1 = -x_1^3 + 2x_1^2 \sin x_1, \quad \dot{x}_2 = x_2 \sin^2 x_2;$

are unstable at the origin.

5. Prove that the differential equations

(a) $\ddot{x} + \dot{x} - \dot{x}^3/3 + x = 0;$ (b) $\ddot{x} + \dot{x} \sin(\dot{x}^2) + x = 0;$

(c) $\ddot{x} + \dot{x} + x^3 = 0;$ (d) $\ddot{x} + \dot{x}^3 + x^3 = 0,$

have asymptotically stable zero solutions $x(t) \equiv 0$.

6. Prove that $V(x_1, x_2) = ax_1^2 + 2bx_1 x_2 + cx_2^2$ is positive definite if and only if $a > 0$ and $ac > b^2$. Hence or otherwise prove that

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 - x_2 + (x_1 + 2x_2)(x_2^2 - 1)$$

is asymptotically stable at the origin by considering the region $|x_2| < 1$. Find a domain of stability.

7. Find domains of stability for the following systems by using the appropriate Liapunov function:

(a) $\dot{x}_1 = x_2 - x_1(1 - x_1^2 - x_2^2)(x_1^2 + x_2^2 + 1)$
$\dot{x}_2 = -x_1 - x_2(1 - x_1^2 - x_2^2)(x_1^2 + x_2^2 + 1);$

(b) $\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_2 + x_2^3 - x_1^5.$

8. Use $V(x_1, x_2) = (x_1/a)^2 + (x_2/b)^2$ to show that the system

$$\dot{x}_1 = x_1(x_1 - a), \qquad \dot{x}_2 = x_2(x_2 - b), \qquad a, b > 0,$$

has an asymptotically stable origin. Show that all trajectories tend to the origin as $t \to \infty$ in the region

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} < 1.$$

9. Given the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_2 - x_1^3$$

show that a positive definite function of the form

$$V(x_1, x_2) = ax_1^4 + bx_1^2 + cx_1x_2 + dx_2^2$$

can be chosen such that $\dot{V}(x_1, x_2)$ is also positive definite. Hence deduce that the origin is unstable.

10. Show that the origin of the system

$$\dot{x}_1 = x_2^2 - x_1^2, \qquad \dot{x}_2 = 2x_1x_2$$

is unstable by using

$$V(x_1, x_2) = 3x_1x_2^2 - x_1^3.$$

Solutions.

1. (d) $\dot{V}(x_1, x_2) = -2x_1^2(\sin x_1)^2 - 2x_2^2 - 2x_2^6$ is negative definite when $x_1^2 + x_2^2 < \pi^2$.

   (e) $\dot{V}(x_1, x_2) = -2x_1^2(1 - x_2) - 2x_2^2(1 - x_1)$ is negative definite when $x_1^2 + x_2^2 < 1$.

2. The domain of stability is $\mathbb{R}^2$ for (a), (b) and (c) and $\{(x_1, x_2) \mid x_1^2 + x_2^2 < r^2\}$ where $r = \pi$ for (d) and $r = 1$ for (e).

3. Asymptotically stable: (a) and (b).
   Neutrally stable: (c) and (d).

4. The system $\dot{\mathbf{x}} = -\mathbf{X}(\mathbf{x})$ has an asymptotically stable fixed point at the origin. Let $\mathbf{x}_0$ be such that $\lim_{t \to \infty} \phi_t(\mathbf{x}_0) = \mathbf{0}$. Choose a neighbourhood N of $\mathbf{0}$ not containing $\mathbf{x}_0$. The trajectory through $\mathbf{x}_0$ of the system $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$ satisfies $\lim_{t \to -\infty} \phi_t(\mathbf{x}_0) = \mathbf{0}$. Use this property to show that the origin is unstable. Use the function $V(x_1, x_2) = x_1^2 + x_2^2$ in (a) to (c).

5. Use the function $V(x_1, x_2) = x_1^2 + x_2^2$ in (a) and (b) and $V(x_1, x_2) = x_1^4 + 2x_2^2$ in (c) and (d).

6. If V is positive definite then $V(1, 0)$ is positive and so a is positive; also

$$V(x_1, x_2) = a\left(x_1 + \frac{b}{a}x_2\right)^2 + \left(c - \frac{b^2}{a}\right)x_2^2$$

and thus a and $c - b^2/a$ are positive. Try $a = 5, b = 1, c = 2$; then

$$V(x_1, x_2) = 5\left(x_1 + \frac{x_2}{5}\right)^2 + \frac{9}{5}x_2^2.$$

For $V(x_1, x_2) < 9/5, x_2^2 < 1$ and so there is a domain of stability defined by $25x_1^2 + 10x_1x_2 + 10x_2^2 < 9$.

7. (a) $V(x_1, x_2) = x_1^2 + x_2^2$, $\dot{V}(x_1, x_2) = -2r^2(1 - r^2)(1 + r^2)$; $x_1^2 + x_2^2 < 1$.

   (b) $V(x_1, x_2) = x_1^6 + 3x_2^2$, $\dot{V} = -x_2^2(1 - x_2^2)$; $x_1^6 + 3x_2^2 < 3$.

8. $\dot{V}(x_1, x_2) = \frac{2x_1^2}{a^2}(x_1 - a) + \frac{2x_2^2}{b^2}(x_2 - b)$ is negative definite for $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} < 1$.

9. $V(x_1, x_2) = \frac{1}{2}x_1^4 + \frac{1}{2}x_1^2 - x_1 x_2 + x_2^2$ satisfies the hypotheses of Theorem 5.4.3.

10. $\dot{V}(x_1, x_2) = 3(x_1^2 + x_2^2)^2$.