

May 27, 2020

1 Banach's contraction principle. Picard-Lindelöf theorem.

We consider in this chapter the theorem by Picard and Lindelöf about existence and uniqueness of solutions to the initial value problem to the system of differential equations in the form

$$x'(t) = f(t, x(t)) \tag{1}$$

$$x(\tau) = \xi \tag{2}$$

Here $f : J \times G \rightarrow \mathbb{R}^n$ is a vector valued function continuous with respect to time variable t and space variable x . J is an interval, G is an open subset of \mathbb{R}^n .

One can reformulate the I.V.P. (1),(2) in the form of the integral equation

$$x(t) = \xi + \int_{\tau}^t f(s, x(s)) ds \tag{3}$$

If f is continuous, then these two formulations are equivalent by the Newton-Leibnitz theorem.

Fixed points of operators.

Consider a vector space X with a subset $C \subset X$ and an operator $K : C \rightarrow C$.

Definition

A point $\bar{x} \in C$ is called the **fixed point** of the operator K if

$$K(\bar{x}) = \bar{x} \tag{4}$$

A general idea behind the analysis of many types of equations is to reformulate them as a fixed point problem.

Consider the right hand side of the integral equation (3) as an operator

$$K(x)(t) \stackrel{\text{def}}{=} \xi + \int_{\tau}^t f(s, x(s)) ds$$

acting from the vector space of continuous functions $C(I)$, where $I \subset J$ is a closed interval including τ . Point out that t can be smaller than τ ($t < \tau$).

The expression $\|x\|_{C(I)} = \sup_{t \in I} \|x(t)\|$ defines a norm on the space $C(I)$ because it satisfies the triangle inequality and we know that uniformly convergent sequences of continuous functions on the compact set (I in this case) converge to continuous functions.

This space is even complete in the sense that Cauchy sequences of functions in $C(I)$ converge uniformly to continuous functions. It means more explicitly that if the sequence $\{x_n\} \in C(I)$ has the Cauchy property:

$$\|x_m - x_n\|_{C(I)} = \sup_{t \in I} \|x_m(t) - x_n(t)\|_{C(I)} \xrightarrow{m, n \rightarrow \infty} 0$$

then there is a continuous function $\bar{x} \in C(I)$ such that $x_n \xrightarrow{n \rightarrow \infty} \bar{x}$ uniformly on I , or that is the same, $\|x_n - \bar{x}\|_{C(I)} \xrightarrow{n \rightarrow \infty} 0$.

Definition.

We call a normed vector space a Banach space if it is complete with

respect to its norm.

This notion was introduced by Polish mathematician Stefan Banach who led the greatest school in functional analysis at the university of Lwow in the beginning of 20th century.

Examples.

The space $C(I)$ is a Banach space.

Elementary examples of Banach spaces are given by \mathbb{R}^n supplied with norms $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ with $p \geq 1$.

A slight extension of this example is a set l_p , $p \geq 1$ of real sequences $\{x_i\}_{i=1}^\infty$ with finite norms in the form $\|x\|_p = (\sum_{i=1}^\infty |x_i|^p)^{1/p}$.

One of the most popular classes of Banach spaces is the space of "integrable functions" $f : G \rightarrow \mathbb{R}$ where $G \subset \mathbb{R}^n$, with norms $\|f\|_{L^p} = (\int_G |f(z)|^p dz)^{1/p}$

"Integrable functions" and the integral here are in the sense of Lebesgue, that is a contemporary notion of integral, studied in the course "Integration theory" given for master and PhD students.

Remark.

We point out for convenience that different norms are used through out the text. Notation $\|\cdot\|$ means usual euclidean norm in \mathbb{R}^n . For a Banach space X the notation $\|x\|_X$ means the norm in the space X .

The operator K defined above, acts from $C(I)$ to itself. It makes that the I.V.P. above can be considered as a fixed value problem (4) on $C(I)$ or on some subset of it.

A classical theorem that guarantees the existence and uniqueness of fixed points to operators in Banach and more generally in metric spaces, is Banach's contraction principle.

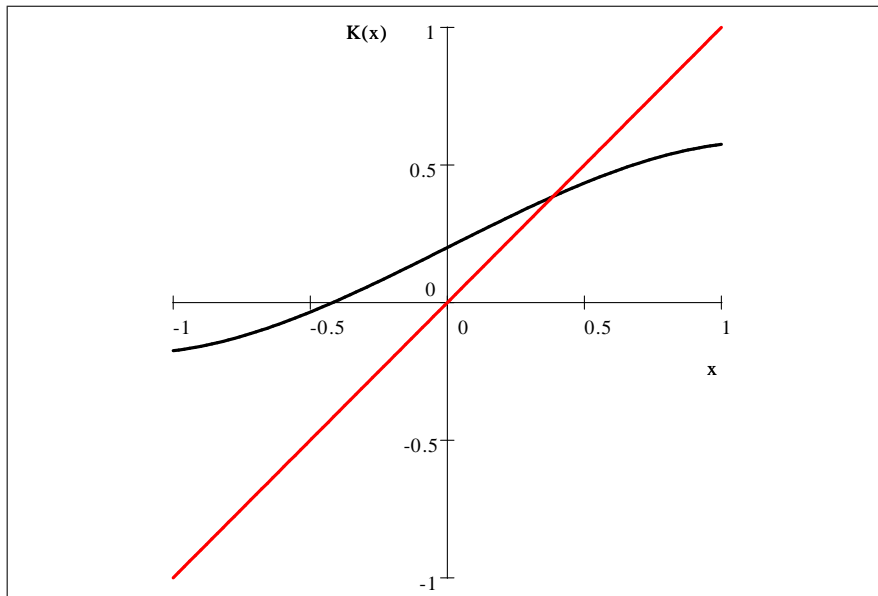
Definition. Operator $K : A \rightarrow A$, where $A \subset X$, and X is a Banach space, is called contraction on A if there is a constant $0 < \theta < 1$ such that for any $x, y \in A$

$$\|K(x) - K(y)\|_X \leq \theta \|x - y\|_X$$

Example. An elementary example is a smooth function K acting from an interval $[a, b]$ to itself and having absolute value of derivative $|\frac{d}{dt}K(t)| < \theta < 1$ for all $t \in [a, b]$. By intermediate value theorem for any $x, y \in [a, b]$ there is a point $c \in (x, y)$ such that $K(x) - K(y) = (x - y)K'(c)$. Therefore

$$|K(x) - K(y)| = |(x - y)| |K'(c)| \leq \theta |(x - y)|$$

It implies that K is a contraction in on the interval $[a, b]$. Example:
 $K(x) = 0.5(x - 0.25x^3) + 0.2$



Another example would be a Lipschitz function with Lipschitz constant L smaller than one: $L < 1$.

Banach's contraction principle.

Let A be a non-empty closed subset of a Banach space X and $K : A \rightarrow A$ be a contraction operator with contraction constant $\theta < 1$.

Then there is a unique fixed point $\bar{x} \in A$, to K such that $K\bar{x} = \bar{x}$ such that

$$\|K^n(x_0) - \bar{x}\|_X \leq \frac{\theta^n}{1 - \theta} \|K(x_0) - \bar{x}\|_X$$

for arbitrary $x_0 \in A$. Here $K^n(x_0) = K(K(\dots K(x_0))\dots)$ is the operator K applied to itself n times.

Proof (not required at the exam) is based on showing that the sequence of approximations $\{x_n\}_{n=0}^\infty$ defined by the equations

$$\begin{aligned} x_1 &= K(x_0) \\ &\dots \\ x_{n+1} &= K(x_n) \end{aligned}$$

with an arbitrary initial approximation $x_0 \in A$, converge to some $\bar{x} \in A$ that is the unique fixed point of K in A .

It follows by induction that

$$\begin{aligned} \|x_{n+1} - x_n\|_X &= \|K(x_n) - K(x_{n-1})\|_X \leq \theta \|x_n - x_{n-1}\|_X \\ &\leq \theta \|K(x_{n-1}) - K(x_{n-2})\|_X \leq \theta^2 \|x_{n-1} - x_{n-2}\|_X \\ &\dots \\ &\leq \theta^n \|x_1 - x_0\|_X \end{aligned}$$

We will show that $\{x_n\}_{n=0}^\infty$ is a Cauchy sequence. Let $m > n$.

$$\begin{aligned}
\|x_m - x_n\|_X &= \|x_m - x_{m-1} + x_{m-1} - x_{m-2} + \dots + x_{n+1} - x_n\|_X \\
&\leq \|x_m - x_{m-1}\| + \|x_{m-1} - x_{m-2}\| + \dots + \|x_{n+1} - x_n\| \leq \\
&\leq (\theta^n + \theta^{n-1} + \dots + \theta^{m-1}) \|x_1 - x_0\|_X \\
&= \theta^n (1 + \theta + \dots + \theta^{m-n-1}) \|x_1 - x_0\|_X \\
&\leq \theta^n (1 + \theta + \dots + \theta^{m-n-1} + \dots) \|x_1 - x_0\|_X \\
&\leq \theta^n \left(\frac{1}{1-\theta} \right) \|x_1 - x_0\|_X \rightarrow 0, \quad n \rightarrow \infty, \quad \theta < 1
\end{aligned}$$

A is closed, therefore the limit $\lim_{n \rightarrow \infty} x_n = \bar{x}$ exists $\bar{x} \in A$.

Claim: \bar{x} is a fixed point to K . We see it by tending to the limit in the expression for $x_n: x_{n+1} = K(x_n)$

$$\begin{aligned}
\lim_{n \rightarrow \infty} x_{n+1} &= \lim_{n \rightarrow \infty} K(x_n) = K(\lim_{n \rightarrow \infty} x_n) \\
\bar{x} &= K(\bar{x})
\end{aligned}$$

and using continuity of K .

The last question we must answer is the uniqueness of the fixed point to K in A . Suppose that here is another fixed point \tilde{x} to K in A . Consider the norm of the difference $\bar{x} - \tilde{x}$:

$$\|\bar{x} - \tilde{x}\|_X = \|K(\bar{x}) - K(\tilde{x})\|_X \leq \theta \|\bar{x} - \tilde{x}\|_X, \quad \theta < 1$$

It is possible only if $\bar{x} - \tilde{x} = 0$.

$$\begin{aligned} \|x_m - x_n\|_X &\leq \theta^n \left(\frac{1}{1-\theta} \right) \|x_1 - x_0\|_X \\ \lim_{m \rightarrow \infty} \|x_m - x_n\|_X &\leq \theta^n \left(\frac{1}{1-\theta} \right) \|x_1 - x_0\|_X \\ \left\| \lim_{m \rightarrow \infty} x_m - x_n \right\|_X &\leq \theta^n \left(\frac{1}{1-\theta} \right) \|x_1 - x_0\|_X \\ \|\bar{x} - x_n\|_X &\leq \theta^n \left(\frac{1}{1-\theta} \right) \|x_1 - x_0\|_X \end{aligned}$$

■

Elementary exercise on Banach's contraction principle.

Show using Banach's contraction principle that the polynomial $K(x) = x^2 - 0.4$ has a fixed point $K(x) = x$.

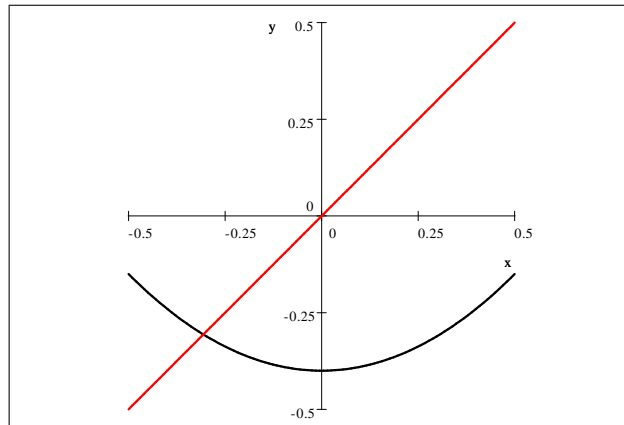
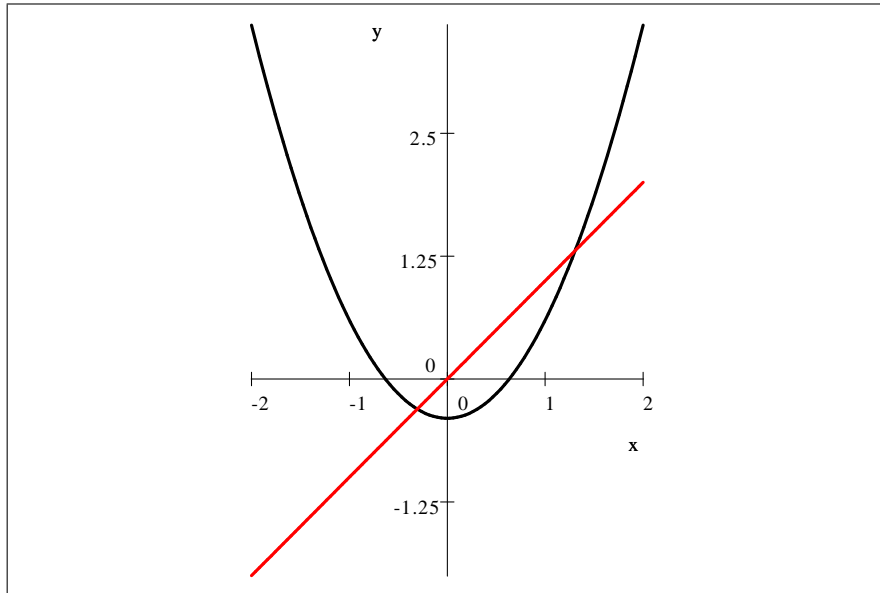
Solution consists of two steps.

i) Find a set $B \subset \mathbb{R}$ where $K(x)$ has the contraction property: $|K(x) - K(y)| \leq \theta |x - y|$, $\theta < 1$, for $x, y \in B$

ii) Find a subset $A \subset B$ that the function K maps into itself: $K : A \rightarrow A$.

i) $K'(x) = 2x < 1 \implies x \in [-0.5 + \delta, 0.5 - \delta]$

ii) The set $[-0.5 + \delta, 0.5 - \delta]$ satisfies the requirement.



Picard-Lindelöf theorem.

Here $f : J \times G \rightarrow \mathbb{R}^n$ is a vector valued function continuous in $J \times G$. J is an interval, G is an open subset of \mathbb{R}^n . Let in addition suppose that f is Lipschitz continuous with respect to the second argument with the Lipschitz constant $L > 0$:

$$\|f(t, x) - f(t, y)\| \leq L \|x - y\|, \forall x, y \in G$$

(We could suppose a weaker condition that this Lipschitz property is only local, but will not do it because it would make the proof just slightly longer without changing main ideas).

Then for any $(\tau, \xi) \in J \times G$ the initial value problem

$$\begin{aligned}x' &= f(x, t) \\x(\tau) &= \xi\end{aligned}$$

has a unique solution on some time interval including τ . \square

Remark. This local solution can always be extended to a unique maximal solution. We considered maximal extensions earlier in the course.

Proof to the Picard-Lindelöf theorem.

The proof is based on using the integral form of the I.V.P.

$$x(t) = \xi + \int_{\tau}^t f(s, x(s)) ds$$

and applying Banach's contraction principle to it. We use the Banach space of continuous functions $x : I \rightarrow \mathbb{R}^n$ on some compact interval $I \subset J$.

The application of Banach's principle here consists of two steps.

- The first one is to find a time interval I_1 and a subset $A \subset C(I_1)$ such that the operator K defined by

$$K(x)(t) = \xi + \int_{\tau}^t f(s, x(s)) ds$$

maps A to itself: $K : A \rightarrow A$.

- The second one is to find a time interval I_2 such that the contractness property for the operator would be valid on a subset of $C(I_2)$. Finally we will choose the smallest of I_1 and I_2 for both properties to be valid and will conclude the result.

We consider here the case when an interval $[\tau, \tau + T] \in J$, $T > 0$ and try to find a solution on this time interval (or possibly on a shorter time interval

$[\tau, \tau + \Delta]$ with $\Delta < T$). Considering a time interval backward direction in time is similar.

We choose first a closed ball $\overline{B(\xi, \delta)} = \{x : \|x - \xi\| \leq \delta\}$ such that it belongs to G : $\overline{B(\xi, \delta)} \in G$.

Our intension is to find solution in the set of continuous functions $x : [\tau, \tau + T] \rightarrow \mathbb{R}^n$ such that $x(t) \in \overline{B(\xi, \delta)}$ for all $t \in [\tau, \tau + T]$ and therefore $\sup_{t \in [\tau, \tau + T]} \|x(t) - \xi\| \leq \delta$. It is a closed ball

$$A = \|x - \xi\|_{C([\tau, \tau + T])} \leq \delta$$

in the infinitely dimensional space $C([\tau, \tau + T])$.

Our goal in the proof is to find such an interval $[\tau, \tau + T]$ that this set A in $C([\tau, \tau + T])$ and the operator K satisfy conditions in the Banach contraction principle.

The function $f(t, x)$ is continuous on the compact set $V = [\tau, \tau + T] \times \overline{B(\xi, \delta)}$ in \mathbb{R}^{n+1} and therefore

$$M = \sup_{(t, x) \in V} \|f(t, x)\| < \infty$$

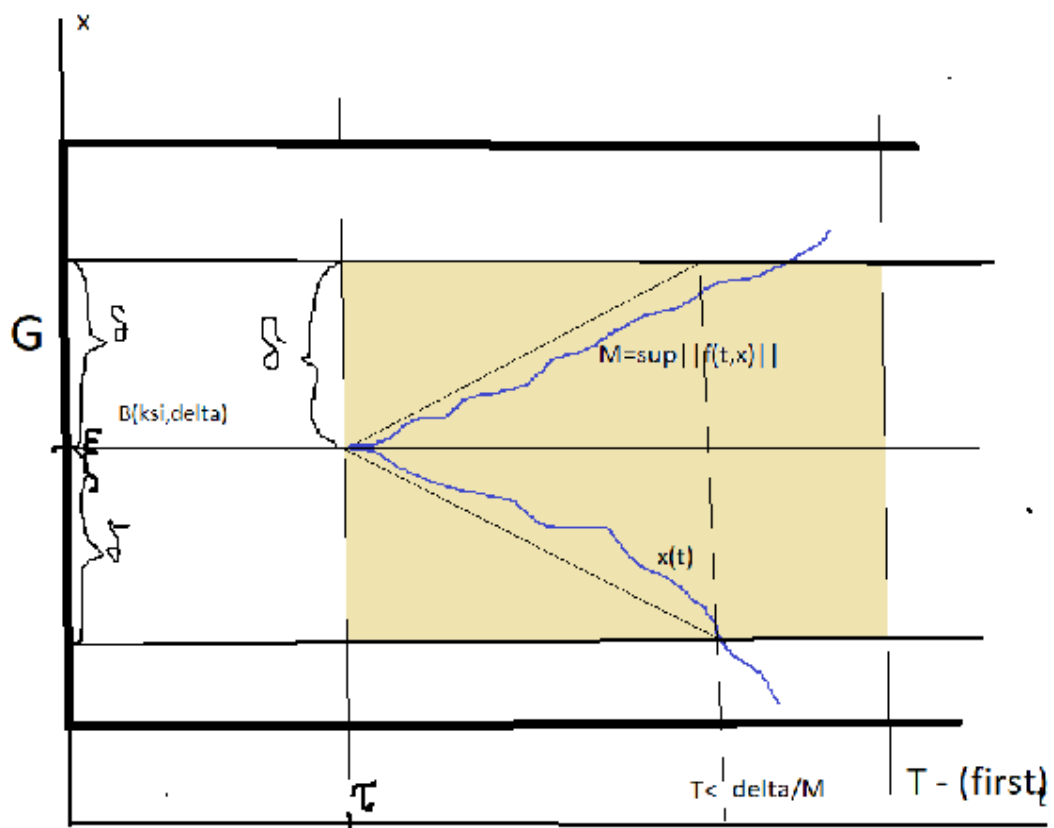
The constant M controls how large is velocity $f(t, x)$ inside the set $V = [\tau, \tau + T] \times \overline{B(\xi, \delta)}$ (yellow in the picture). Correspondingly M controls how fast the (blue) trajectory $x(t)$ can go away from the initial point ξ .

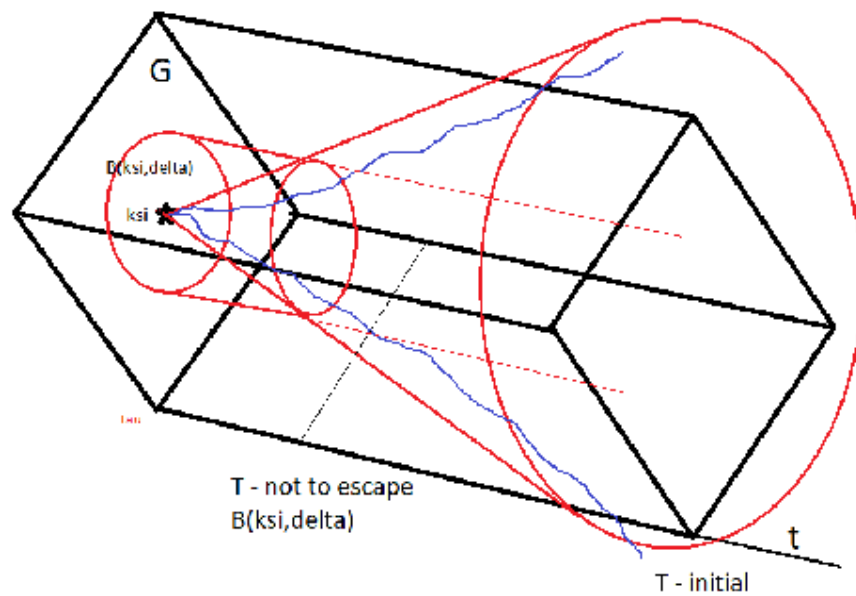
According to the equation $x(t) = K(x)$ must be inside the "angle" bounded by the cone $\|x - \xi\| = M(t - \tau)$.

We give here two pictures illustrating the proof, a one dimensional picture: and a two-dimensional picture:

We are going to estimate $\|K(x)(t) - \xi\|$ and choose the length T of the time interval $[\tau, \tau + T]$ in such a way that for any $x(t) \in \overline{B(\xi, \delta)}$ for $t \in [\tau, \tau + T]$, it follows that $K(x(t))$ does not escape the ball $\overline{B(\xi, \delta)}$.

$$\|K(x(t)) - \xi\| \leq \delta$$





for $t \in [\tau, \tau + T]$.

It would imply that

$$\sup_{t \in [\tau, \tau + T]} \|K(x)(t) - \xi\| = \|K(x) - \xi\|_{C([\tau, \tau + T])} \leq \delta$$

for $\|x - \xi\|_{C([\tau, \tau + T])} \leq \delta$. We start with proving the first inequality:

$$\|K(x)(t) - \xi\| = \left\| \int_{\tau}^t f(s, x(s)) ds \right\| \leq \int_{\tau}^t \|f(s, x(s))\| ds \leq TM$$

We observe that choosing $T < \delta/M$ we get that $\|K(x)(t) - \xi\| \leq \delta$ for $t \in [\tau, \tau + T]$. Taking supremum of the left hand side over $t \in [\tau, \tau + T]$ arrive to

$$\|K(x) - \xi\|_{C([\tau, \tau + T])} \leq \delta$$

It means that for this time interval trajectories do not gun out of the yellow domain on the picture. In turn it means that the operator K maps the closed ball A in $C([\tau, \tau + T])$ defined by the inequality $\|x - \xi\|_{C([\tau, \tau + T])} \leq \delta$, with

$$T < \delta/M$$

into itself:

$$K : A \rightarrow A$$

Now we check conditions (again choosing the length of the time interval) such that the operator K would be contraction on the set A with once again suitably adjusted time interval T . Consider first the difference $\|K(x)(t) - K(y)(t)\|$, for arbitrary $t \in [\tau, \tau + T]$. We apply the triangle inequality, the Lipschitz property of the function f , and estimate the integral by the length of the interval times maximum of the function under it.

$$\begin{aligned}
\|K(x)(t) - K(y)(t)\| &= \left\| \int_{\tau}^t f(s, x(s)) - f(s, y(s)) ds \right\| \stackrel{\text{triangle inequality}}{\leq} \\
&\leq \int_{\tau}^t \|f(s, x(s)) - f(s, y(s))\| ds \\
\stackrel{\text{Lipschitz property}}{\leq} &L \int_{\tau}^t \|x(s) - y(s)\| ds \leq \\
&\stackrel{\sup}{\leq} LT \sup_{s \in [\tau, \tau+T]} \|x(s) - y(s)\| = LT \|x - y\|_{C([\tau, \tau+T])}
\end{aligned}$$

Calculating supremum over $t \in [\tau, \tau + T]$ of the left hand side we arrive to the inequality

$$\|K(x) - K(y)\|_{C([\tau, \tau+T])} \leq LT \|x - y\|_{C([\tau, \tau+T])}$$

It implies that choosing the length of the time interval $T < 1/L$ we get the contraction property.

$$\|K(x) - K(y)\|_{C([\tau, \tau+T])} \leq \theta \|x - y\|_{C([\tau, \tau+T])}, \quad 0 < \theta < 1$$

Now choosing the time interval $T < \min(1/L, \delta/M)$ we conclude that the operator K maps the closed ball A in $C([\tau, \tau + T])$ defined by

$$A = \left\{ x \in C([\tau, \tau + T]), \|x - \xi\|_{C([\tau, \tau+T])} \leq \delta \right\}$$

into itself: $K : A \rightarrow A$ and that K is a contraction on A : $\|K(x) - K(y)\|_{C([\tau, \tau+T])} \leq \theta \|x - y\|_{C([\tau, \tau+T])}$, $\theta < 1$, for any $x, y \in A$.

By the Banach contraction principle K has for $T < \min(1/L, \delta/M)$ a unique fixed point \bar{x} in A that is the solution to the integral equation (3) and to the original initial value problem. ■

Example. Banach's contraction principle applied to a non-linear integral operator. (exam 2019 june)

Consider the following (nonlinear!) operator

$$K(x)(t) = \int_0^2 B(t, s) [x(s)]^2 ds + g(t),$$

Fixed point problem to solve:

$$x = K(x)$$

acting on the Banach space $C([0, 2])$ of continuous functions with norm $\|x\|_{C([0,2])} = \|x\|_C = \sup_{t \in [0,2]} |x(t)|$. Here $B(t, s)$ and $g(t)$ are continuous functions and $|B(t, s)| < 0.5$ for all $t, s \in [0, 2]$. Estimate the norm $\|K(x) - K(y)\|_{C([0,2])}$ for the operator $K(x)(t)$. Find requirements on the function $g(t)$ such that Banach's contraction principle implies that $K(x)(t)$ has a fixed point.

Solution.

Banach's contraction principle. Let B be a nonempty closed subset of a Banach space X and let the non-linear operator $K : B \rightarrow B$ be a contraction.

$$\|K(x) - K(y)\|_X \leq \theta \|x - y\|_X, \quad \theta < 1$$

Then K has a fixed point $\bar{x} = K(\bar{x})$ such that

$$\|K^n(x_0) - \bar{x}\|_X \leq \frac{\theta^n}{1 - \theta}$$

for any $x_0 \in B$. Here $K^n(x_0) = (K(K(\dots K(x_0)\dots)))$ is the n -fold superposition of the operator K with itself.

We like to have the estimate $\|K(x) - K(y)\| \leq \theta \|x - y\|$ for x, y in some closed subset B of $C([0, 2])$.

$$\begin{aligned}
|K(x)(t) - K(y)(t)| &\leq \left| \int_0^2 |B(t,s)| |[x(s)]^2 - [y(s)]^2| ds \right| \\
&= \left| \int_0^2 |B(t,s)| \cdot |x(s) - y(s)| \cdot |x(s) + y(s)| ds \right| \stackrel{\text{taking}}{\leq} \sup_{t,s \in [0,2]} \\
&\leq \int_0^2 ds \sup_{t,s \in [0,2]} |B(t,s)| \sup_{s \in [0,2]} |x(s) - y(s)| \left(\sup_{s \in [0,2]} |x(s)| + \sup_{s \in [0,2]} |y(s)| \right) = \\
&= 2 \cdot 0.5 \|x - y\|_C (\|x\|_C + \|y\|_C) = \|x - y\|_C (\|x\|_C + \|y\|_C)
\end{aligned}$$

We take supremum over $t \in [0, 2]$ of the left hand side and get

$$\|K(x) - K(y)\|_C \leq \|x - y\|_C (\|x\|_C + \|y\|_C)$$

We can choose a ball $B \subset C([0, 2])$ such that for any $x, y \in B$ it follows $\|x\|_C + \|y\|_C \leq \theta < 1$, for example B can be taken as a set of continuous functions with $\|x\|_C \leq \theta/2$. On this set K will be a contraction because

$$\|K(x) - K(y)\|_C \leq \theta \|x - y\|_C, \quad \theta < 1.$$

To apply Banach's principle we need also that K maps B into itself, namely that $\|K(x)\|_C \leq \theta/2$ for $\|x\|_C \leq \theta/2$.

It gives a requirement on function $g(t)$.

$$\begin{aligned}
K(x)(t) &= \int_0^2 B(t,s) [x(s)]^2 ds + g(t), \\
\|K(x)\|_C &\leq 2 \times 0.5 \times \|x\|_C^2 + \|g\|_C \leq (\theta/2)^2 + \|g\|_C \leq \theta/2
\end{aligned}$$

Conclusion is that $\|g\|_C = \sup_{t \in [0,2]} |g(t)| \leq \theta/2 - (\theta/2)^2 = \theta/2(1 - \theta/2)$ implies that $K : B \rightarrow B$, where $B = \{x(t) : |x(t)| \leq \theta/2\}$, $x(t)$ - continuous.

Therefore K has a unique fixed point in the ball B in $C([0, 2])$. ■

Example. (exam. 2018 january)

1. Consider the following initial value problem: $y' = \sin(y)t^2$; $y(1) = 2$.
 - a) Reduce the initial value problem to an integral equation and give a general description of iterations approximating the solution as in the proof to the existence and uniqueness theorem by Picard and Lindelöf. **(2p)**
 - b) Find a time interval such that these approximations converge to the solution of the initial value problem. **(2p)**

Solution.

We introduce an integral equation equivalent to the ODE $y' = f(t, y)$ by the integration of the right and left hand sides in the equation:

$$y(t) = y(1) + \int_1^t f(s, y(s)) ds.$$

Taking $y_0(t) = y(1)$ we define Picard iterations by the recurrence relation

$$\begin{aligned} y_{n+1}(t) &= y(1) + \int_1^t f(s, y_n(s)) ds. \\ y_{n+1} &= \mathbb{K}(y_n) \end{aligned}$$

For the particular equation it looks as

$$y_{n+1}(t) = y(1) + \int_1^t \sin(y_n(s)) s^2 ds = \mathbb{K}(y_n, t).$$

Fixed point problem:

$$y = \mathbb{K}(y)$$

The Banach contraction principle gives existence and uniqueness of

solutions by showing that the operator \mathbb{K} is a contraction on some closed set B of a Banach space X , such that \mathbb{K} maps B into itself.

A hidden question here is that we must find this Banach space X and this set B where these conditions are satisfied.

One proves the existence and uniqueness theorem by showing that at some time interval the integral operator $\mathbb{K}(y, t) = y(1) + \int_1^t \sin(y(s))s^2 ds$ in the right hand side is a contraction in $C([1, T])$:

$$\|\mathbb{K}(w) - \mathbb{K}(u)\|_{C([1, T])} \stackrel{def}{=} \sup_{t \in [1, T]} |\mathbb{K}(w, t) - \mathbb{K}(u, t)| < \alpha \sup_{t \in [1, T]} |w(t) - u(t)| = \alpha \|w - u\|_{C([1, T])}$$

$\alpha < 1$, in some ball $\|w - y(1)\|_{C([1, T])} = \sup_{t \in [1, T]} |w(t) - y(1)| \leq R$ in the space $C([1, T])$ of continuous functions on $[1, T]$, and maps this ball into itself:

$$\sup_{t \in [1, T]} |\mathbb{K}(w, t) - y(1)| \leq R$$

and applying the Banach contraction theorem to $\mathbb{K}(y, t)$.

We estimate first $\|\mathbb{K}(w) - \mathbb{K}(u)\|_{C([1, T])} = \sup_{t \in [1, T]} |\mathbb{K}(w, t) - \mathbb{K}(u, t)|$ for continuous functions u and w such that $\sup_{t \in [1, T]} |w(t) - y(1)| \leq R$ and

$\|w - y(1)\|_{C([1, T])} = \sup_{t \in [1, T]} |u(t) - y(1)| \leq R$. Point out that $\sup_{t \in [1, T]} |w(t)| \leq y(1) + R$. We will find T such that the contraction property is valid:

$$\|\mathbb{K}(w) - \mathbb{K}(u)\|_{C([1, T])} = \sup_{t \in [1, T]} \left| \int_1^t \sin(w(s))s^2 ds - \int_1^t \sin(u(s))s^2 ds \right| \leq \alpha \sup_{t \in [1, T]} |w(t) - u(t)|,$$

We carry out elementary estimates using the triangle inequality and intermediate value theorem for sin. $\left| \int_1^t \sin(w(s))s^2 ds - \int_1^t \sin(u(s))s^2 ds \right| \leq \int_1^t |(\sin(w(s)) - \sin(u(s)))| s^2 ds =$

$$\int_1^t |(w(s) - u(s)) \cos(\theta(s))| s^2 ds \leq (T - 1) T^2 \cdot 1 \cdot \sup_{t \in [1, T]} |w(s) - u(s)|$$

$$\|\mathbb{K}(w) - \mathbb{K}(u)\|_{C([1, T])} \leq (T - 1) T^2 \|w(s) - u(s)\|_{C([1, T])}$$

The argument $\theta(s)$ above is a number between $w(s)$ and $u(s)$ that exists according to the intermediate value theorem. It was also used above that $|\cos(\theta)| \leq 1$. Therefore to have the contraction property we need to have $(T - 1) T^2 < 1$.

For a function w with $\|w(s)\|_{C([1, T])} = \sup_{t \in [1, T]} |w(t) - y(1)| \leq R$ we like to have that $|\mathbb{K}(w, t) - y(1)| \leq R$ meaning that \mathbb{K} maps this ball in $C([1, T])$ into itself. For this particular case it is not necessary because the equation is defined in the whole \mathbb{R} and the contraction property is valid in the whole $C([1, T])$. But this checking might be necessary if the contraction property is valid only locally, not in the whole $C([1, T])$.

The following estimate leads to another bound for T : $\sup_{t \in [1, T]} |\mathbb{K}(w, t) - y(1)| \leq$

$$\sup_{t \in [1, T]} \left| \int_1^t \sin(w(s)) s^2 ds \right| \leq (T - 1) T^2 \leq R.$$

Therefore the time interval must satisfy estimates $(T - 1) T^2 < 1$ and $(T - 1) T^2 < R$ to have convergence of Picard iterations in the ball $\sup_{t \in [1, T]} |w(t) - y(0)| \leq R$. Taking $R = 1$ we get an optimal estimate $(T - 1) T^2 < 1$ that is satisfied for example for $T = 1.4$:

$$\alpha = 0.4(1.4)(1.4) = 0.784$$

Introduction to bifurcations.

Considering differential equations where the right hand side includes a parameter:

$$x' = f(t, x, \mu)$$

we can observe qualitative changes in the phase portrait of the system at certain values of the parameter $\mu = \mu_0$.

Examples of bifurcations.

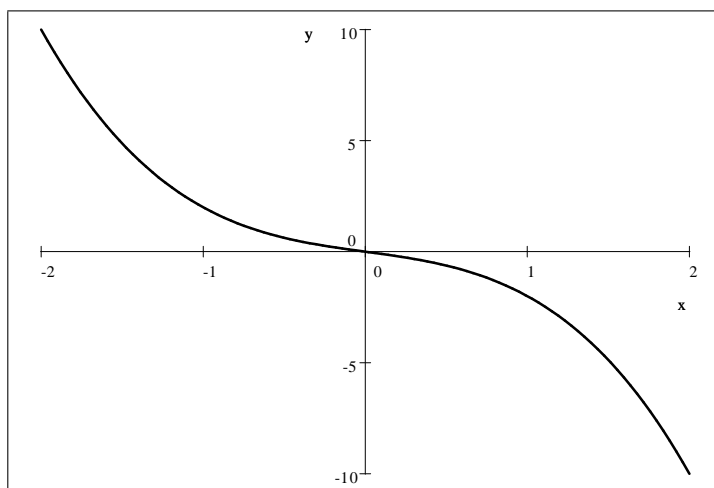
Pitchfork bifurcation

The equation

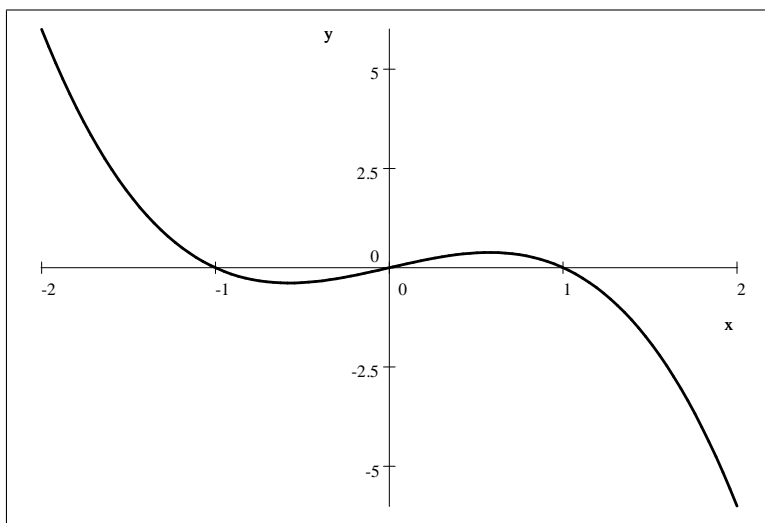
$$x' = \mu x - x^3$$

has one stable equilibrium point $x = 0$ for $\mu \leq 0$, that becomes unstable and splits into two stable equilibrium points at $\mu = 0$.

$$f(x) = -x - x^3, \mu < 0$$



$$f(x) = x - x^3, \mu > 0$$



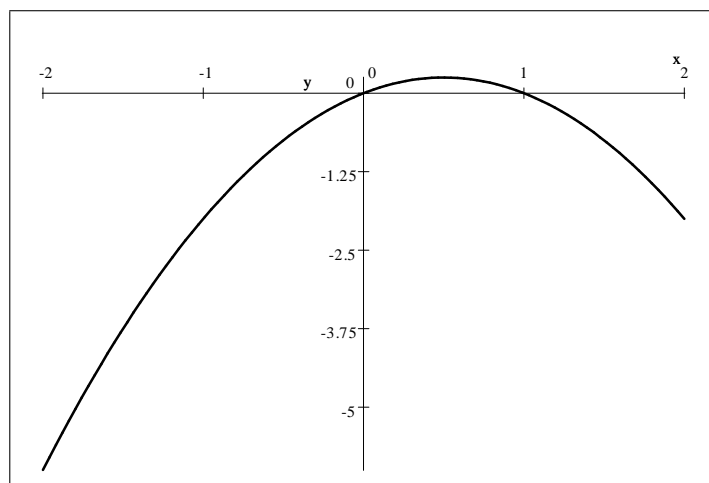
Transcritical bifurcation.

The equation

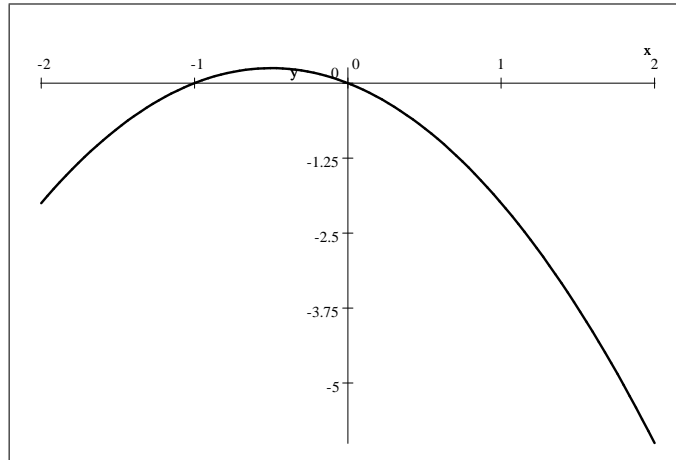
$$x' = \mu x - x^2$$

has two fixed points for $\mu \neq 0$ which collide and exchange stability at $\mu = 0$.

$$f(x) = x - x^2, \mu = 1$$



$$f(x) = -x - x^2, \mu = -1$$



Saddle point bifurcation.

The equation

$$x' = \mu + x^2$$

has one stable and one unstable equilibrium point for $\mu < 0$ which collide at $\mu = 0$ and vanish when $\mu > 0$.

Hopf bifurcation

One impressive example is the so called Hopf bifurcation where an asymptotically stable equilibrium becomes unstable equilibrium surrounded by a unique limit cycle, a periodic solution attracting surrounding trajectories.

The theorem blow gives a possibility to show the existence of a **unique periodic solution** surrounding an equilibrium that is a repeller.

Theorem on Hopf bifurcation. Let the system of differential equations in plane:

$$\begin{aligned} x_1' &= f_1(x_1, x_2, \mu) \\ x_2' &= f_2(x_1, x_2, \mu) \end{aligned}$$

have an equilibrium point in the origin for all real values of the parameter μ .

Suppose that for the linearized system of equation around the origin eigenvalues are purely imaginary for $\mu = \mu_0$. Suppose also that for real part part

of eigenvalues $\operatorname{Re}(\lambda_1(\mu)) = \operatorname{Re}(\lambda_2(\mu))$ the condition

$$\frac{d}{d\mu} \{\operatorname{Re}(\lambda_1(\mu))\}|_{\mu=\mu_0} > 0$$

and that the origin is asymptotically stable for $\mu = \mu_0$.

Then

- i) $\mu = \mu_0$ is a bifurcation point for the system
- ii) there is an interval (μ_1, μ_0) such that the origin is a stable spiral(focus)
- iii) there is an interval (μ_0, μ_2) such that the origin is an unstable spiral(focus), surrounded by a limit cycle (periodic orbit) with size increasing with increasing of μ .

Example. Show that the following system undergoes Hopf bifurcation at $\mu = 0$.

$$\begin{aligned} x_1' &= \mu x_1 - 2x_2 - 2x_1(x_1^2 + x_2^2)^2 \\ x_2' &= 2x_1 + \mu x_2 - x_2(x_1^2 + x_2^2)^2 \end{aligned}$$

Linearized equations are the following:

$$\begin{aligned} x_1' &= \mu x_1 - 2x_2 \\ x_2' &= 2x_1 - \mu x_2 \end{aligned}$$

with matrix $\begin{bmatrix} \mu & -2 \\ 2 & \mu \end{bmatrix}$ with eigenvalues $\lambda_{1,2}(\mu) = \mu \pm 2i$. Therefore $\lambda_{1,2}(0) = \pm 2i$ are purely imaginary.

$\operatorname{Re} \lambda(\mu) = \mu$. and $\frac{d}{d\mu} \operatorname{Re} \lambda(\mu) = 1 > 0$.

The system has a strong Lyapunov function $V(x_1, x_2) = x_1^2 + x_2^2$ for $\mu = 0$.

$$V_f(x_1, x_2) = -2(2x_1^2 + x_2^2)(x_1^2 + x_2^2)^2 < 0, (x_1, x_2) \neq (0, 0)$$

that makes the origin asymptotically stable for $\mu = 0$. Then according to

the Hopf theorem the system undergoes a bifurcation at $\mu = 0$ and at some small $\mu > 0$ it has instable spiral in the origin, surrounded by a periodic orbit. If it is difficult to find a strong Lyapunov function, one can apply LaSalle's invariance principle.

Exercise.

Show that the equation $x'' + (x^2 - \mu)x' + 2x + x^3 = 0$ has a Hopf bifurcation at $\mu = 0$.

Bifurcations will not be at the exam!!!