

1 Bendixson's criterium for nonexistence of periodic solutions in plane.

Theorem. Let $x' = f(x)$ with $f : G \rightarrow \mathbb{R}^2$, $G \subset \mathbb{R}^2$ be open, $f \in C^1(G)$, and let $D \subset G$ be a **simply connected domain** (domain without "holes" even without point holes). It is enough to require that f is locally Lipschitz in G with more knowledge of integration theory.

Suppose that $\operatorname{div}(f) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$ is strictly positive (or strictly negative) in D , where $f = [f_1, f_2]^T$.

Then the equation has no periodic solutions with orbits inside D .

Proof 1. Carry out a proof by contradiction. Suppose that there is a periodic trajectory $x(t)$ with period $T > 0$ in D . $x(t + T) = x(t)$ and

$$x'_1(t) = f_1(x(t)), \quad x'_2(t) = f_2(x(t))$$

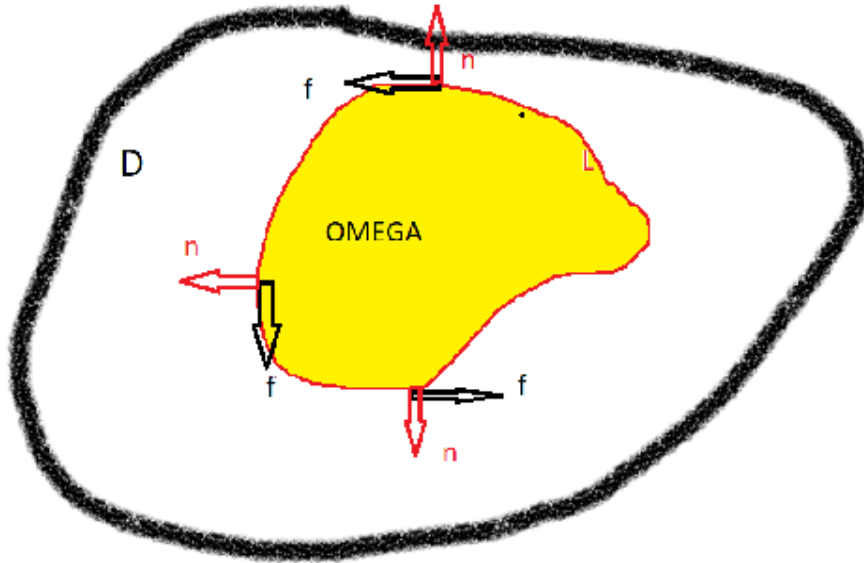
Denote orbit of $x(t)$ by $\mathcal{L} = \{x(t) : t \in [0, T]\}$. It will be a closed curve. Denote the domain inside \mathcal{L} by Ω . Then the boundary $\partial\Omega = \mathcal{L}$ because $D \supset \Omega$ is simply connected and has no holes. Consider the integral of $\operatorname{div}(f)$ over Ω and apply Gauss theorem:

$$I = \int_{\Omega} \operatorname{div}(f) dx_1 dx_2 = \int_{\partial\Omega} f \cdot n dl$$

where n is the outward normal to the boundary $\partial\Omega$. Point out that $f(x(t)) = x'(t)$ on $\partial\Omega = \mathcal{L}$ because \mathcal{L} is the orbit of the periodic solution $x(t)$ that we supposed to be existing. Therefore $f(x(t))$ is the tangent vector to $\partial\Omega$ and therefore scalar product of it with the normal vector is zero $f \cdot n = 0$. Therefore

$$I = \iint_{\Omega} \operatorname{div}(f) dx_1 dx_2 = \int_{\partial\Omega} f \cdot n dl = 0$$

with the curve integral over $\partial\Omega = \mathcal{L}$ in the right hand side. On the other hand $\operatorname{div}(f) > 0$ (or strictly negative) in the whole $D \supset \Omega$. Therefore the integral $I = \int_{\Omega} \operatorname{div}(f) dx_1 dx_2$ over a bounded domain Ω must be strictly positive (negative). We arrived to a contradiction: $0 > 0$. Therefore our supposition



was wrong and the system cannot have a periodic orbit in D . ■

Proof 2. starts similarly with the supposition that there is a periodic trajectory $x(t)$ with period T in D , $x(t+T) = x(t)$ and

$$x'_1(t) = f_1(x(t)), \quad x'_2(t) = f_2(x(t))$$

Denote the orbit of $x(t)$: by $\mathcal{L} = \{x(t) : t \in [0, T]\}$. Denote the domain inside \mathcal{L} by Ω . Then the boundary $\partial\Omega = \mathcal{L}$ because $D \supset \Omega$ is simply connected and has no holes.

We apply the Greens formula:

$$\oint_{\partial\Omega} Pdx_1 + Qdx_2 = \iint_{\Omega} \left(\frac{\partial Q}{\partial x_1} - \frac{\partial P}{\partial x_2} \right) dx_1 dx_2$$

instead of Gauss theorem.

Choose $P = -f_2$, $Q = f_1$ and express the contour integral in the left side

of the Greens formula using the definition of the contour integral:

$$\oint_{\partial\Omega} f_1 dx_2 - f_2 dx_1 = \int_0^T (f_1 x'_2 - f_2 x'_1) dt$$

Point out that $x'_1(t) = f_1(x(t))$ and $x'_2(t) = f_2(x(t))$ and substitute these expressions into the integral:

$$\oint_{\partial\Omega} f_1 dx_2 - f_2 dx_1 = \int_0^T (f_1 f_2 - f_2 f_1) dt = 0$$

Apply the Greens formula substitute expressions for P and Q , and conclude that in the case $\left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right) = \operatorname{div}(f) > 0$:

$$0 = \oint_{\partial\Omega} f_1 dx_2 - f_2 dx_1 = \iint_{\Omega} \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right) dx_1 dx_2 > 0$$

that is contradiction: $0 > 0$. In the case if $\operatorname{div}(f) < 0$ in D we arrive to the contradiction < 0 . ■

1. **Example.**

Show that the following system of ODE has no periodic solutions.

$$1. \quad \begin{cases} x' = x^3 - y^2x + x \\ y' = -0.5y + y^3 + x^4y \end{cases}$$

We consider divergence of the right hand side of the system.

$$\operatorname{div}(f) = 3x^2 - y^2 + 1 - 0.5 + 3y^2 + x^4 = x^4 + 3x^2 + 2y^2 + 0.5 > 0$$

Therefore divergence of the right hand side of the equation is positive everywhere in the plane that is a simply connected set (does not have holes, even point-holes). According to Bendixson's criterium the system cannot have periodic solutions anywhere in the plane.

Example.

Show that the following system of ODE has no periodic solutions.

$$1. \quad \begin{cases} x' = \frac{1}{7} + x^2 - yx + y^2 \\ y' = -\frac{1}{5} - y^2 \end{cases}$$

Solution

y' is always strictly negative. It implies that $y(t)$ must be monotone function of time. It contradicts to possibility of having periodic solutions that are always bounded.