A general way to calculate exponents of matrices. (particularly useful for matrices having complex eigenvalues)

We use here general solution to the equation $x' = Ax$.

We clarify first in which way it can be used.

- For any matrix B the product Be_k gives the column k in the matrix B.
- Therefore the column k in $\exp(A)$ is the product $\exp(A)e_k$, where vector e_k is a standard basis vector, or colum with index k from the unit matrix I.
- On the other hand $\exp(At)\xi$ is a solution to the equation $x' = Ax$ with initial condition $x(0) = \xi$
- The expressions $x_k(t) = \exp(At)e_k$ is a solution to the equation $x' = Ax$ with initial condition $x(0) = e_k$
- Therefore the value of the solution in time $t = 1$: $x_k(1) = \exp(A)e_k$ gives the column k in the matrix $\exp(A)$
- Having the general solution for example in the case of dimension 3:

  $$x(t) = C_1\Psi_1(t) + C_2\Psi_2(t) + C_3\Psi_3(t)$$

  in terms of linearly independent solutions $\Psi_1(t)$, $\Psi_2(t)$, $\Psi_3(t)$, we can for every k find a set of constants $C_{1,k}$,$C_{2,k}$,$C_{3,k}$, corresponding to each of the initial data e_k. Namely we solve equations $C_{1,k}\Psi_1(0)+C_{2,k}\Psi_2(0)+ C_{3,k}\Psi_3(0) = e_k$, $\quad k = 1, 2, 3$
- that are equivalent to the matrix equation

  $$[\Psi_1(0), \Psi_2(0), \Psi_3(0)] \begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{bmatrix} = [e_1, e_2, e_3] = I$$

- Values at $t = 1$ of corresponding solutions:

  $$x_k(1) = C_{1,k}\Psi_1(1) + C_{2,k}\Psi_2(1) + C_{3,k}\Psi_3(1) = \exp(1 \cdot A)e_k$$

  will give us columns $\exp(1 \cdot A)e_k$ in $\exp(A)$.

- In the matrix form this result can be expressed as

$$\begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{bmatrix} = [\Psi_1(0), \Psi_2(0), \Psi_3(0)]^{-1}$$

$$\begin{aligned} \exp(A) &= [\Psi_1(1), \Psi_2(1), \Psi_3(1)] \begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{bmatrix} \\ &= [\Psi_1(1), \Psi_2(1), \Psi_3(1)]\,[\Psi_1(0), \Psi_2(0), \Psi_3(0)]^{-1} \end{aligned}$$

We demonstrate this idea using the result on the general solution from the problem 859.

We can calculate $\exp\left(\begin{bmatrix} 3 & -3 & 1 \\ 3 & -2 & 2 \\ -1 & 2 & 0 \end{bmatrix}\right)$, eigenvalues: $\lambda_1 = -1$, $\lambda_2 = 1 - i$, $\lambda_3 = 1 + i$

General solution to the system $x' = Ax$ is:

$$\begin{aligned} x(t) &= C_1\Psi_1(t) + C_2\Psi_2(t) + C_3\Psi_3(t) \\ &= C_1 e^{-t} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + C_2 e^{t} \begin{bmatrix} \cos t - \sin t \\ \cos t \\ \sin t \end{bmatrix} + C_3 e^{t} \begin{bmatrix} \cos t + \sin t \\ \sin t \\ -\cos t \end{bmatrix} \end{aligned}$$

introducing shorter notations for each term: $x(t) = C_1\Psi_1(t) + C_2\Psi_3(t) + C_3\Psi_3(t)$.

We calculate initial data for arbitrary solution by

$$x(0) = C_1\Psi_1(0) + C_2\Psi_3(0) + C_3\Psi_3(0) = C_1 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + C_3 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$x(0) = [\Psi_1(0), \Psi_3(0), \Psi_3(0)] \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$$

$\exp(A)$ has columns that are values of $x(1)$ for solutions that satisfy initial conditions $r(0) = e_1,\ e_2, e_3$ and therefore $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} C_{1,1} \\ C_{2,1} \\ C_{3,1} \end{bmatrix} =$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = e_1; \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} C_{1,2} \\ C_{2,2} \\ C_{3,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = e_2; \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} C_{1,3} \\ C_{2,3} \\ C_{3,3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = e_3;$$

We solve all three of these systems for $\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$ in one step as a matrix equation

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{bmatrix} = I$$

It is equivalent to the Gauss elimination of the following extended matrix: $\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}$. The result at the rigth half will be the inverted matrix:

$$\begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

It can also found by applying Cramer's rule.

We arrive to the expression of the matrix exponent by collecting these results through the matrix multiplication:

$$\exp(At) = [\Psi_1(t), \Psi_2(t), \Psi_3(t)] \begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{bmatrix}$$

$$\exp(At) = \begin{bmatrix} e^{-t} & e^t(\cos t - \sin t) & e^t(\cos t + \sin t) \\ e^{-t} & e^t \cos t & e^t \sin t \\ -e^{-t} & e^t \sin t & -e^t \cos t \end{bmatrix} \begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} e^t(\cos t + \sin t) - e^{-t} + e^t(\cos t - \sin t) & -e^t(\cos t + \sin t) + e^{-t} & -e^{-t} + e^t(\cos t - \sin t) \\ (\cos t)e^t + (\sin t)e^t - e^{-t} & -(\sin t)e^t + e^{-t} & (\cos t)e^t - e^{-t} \\ -(\cos t)e^t + (\sin t)e^t + e^{-t} & (\cos t)e^t - e^{-t} & (\sin t)e^t + e^{-t} \end{bmatrix}$$

and finally for $t = 1$ we get $\exp(A)$

$$\exp(A) = e \begin{bmatrix} (\cos 1 + \sin 1) - e^{-2} + (\cos 1 - \sin 1) & -(\cos 1 + \sin 1) + e^{-2} & -e^{-2} + (\cos 1 - \sin 1) \\ (\cos 1) + (\sin 1) - e^{-2} & -(\sin 1) + e^{-2} & (\cos 1) - e^{-2} \\ -(\cos 1) + (\sin 1) + e^{-2} & (\cos 1) - e^{-2} & (\sin 1) + e^{-2} \end{bmatrix}$$