1 General properties of ω -limit sets and La Salle's invariance principle and it's applications to asymptotic stability §5.2

Example. An elementary introduction to LaSalle's invariance principle.

We like to investigate stability of equilibrium poin tin the origin for the system

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 - x_2^3 \end{aligned}$$

Using the simple test function $V(x_1, x_2) = x_1^2 + x_2^2$ we observe that it is a Lyapunov function for the system:

$$V_f(x_1, x_2) = \nabla V \cdot f(x_1, x_2) = 2x_1x_2 - 2x_1x_2 - 2x_2^4 = -2x_2^4 \leq 0$$

and the origin is a stable equilibrium point. On the other hand V is not a strong Lyapunov function, because $V_f(x_1, x_2) = 0$ not only in the origin, but on the whole x_1 - axis where x_2 is zero.

On the other hand considering the vector field of velocities of this system on the x_1 - axis, we observe that they are crossing the x_1 - axis (even are orthogonal to it in this particular example) in all points except the origin. It means that all trajectories of the system cross and immediately leave the x_1 - axis that is the line where $V_f(x_1, x_2) = 0$ (the Lyapunov function is not strong). This observation shows that in fact the Lyapunov function $V(\varphi(t, \xi))$ is strictly monotone along trajectories $\varphi(t, \xi)$ everywhere except discret time moments, when $\varphi(t, \xi)$ crosses the x_1 - axis. More explicitely in polar coordinates r and θ:

$$\left(r^2\right)' = -2r^4 \sin^4 \theta$$

We can therefore conclude that $V(\varphi(t, \xi)) \searrow 0$ as $t \to \infty$ and therefore, the origin is asymptotically stable equilibrium of this system of equations.

One can also get a more explicit picture of this dynamics by looking on the equation for the polar angle θ:

$$\begin{aligned}
\left(\frac{x_2}{x_1}\right)' &= (\tan(\theta))' = \frac{\theta'}{\cos^2(\theta)} \\
\frac{x_2' x_1 - x_1' x_2}{x_1^2} &= \frac{(-x_1 - x_2^3)\, x_1 - (x_2)\, x_2}{x_1^2} \\
&= \frac{(-x_1^2 - x_2^2 - x_1 x_2^3)}{x_1^2} = \frac{-r^2 - \cos\theta \sin^3\theta \, r^4}{r^2 \cos^2\theta}
\end{aligned}$$

$$\begin{aligned}
\theta' &= -1 - \cos\theta \sin^3\theta \, r^2 = -1 - \frac{\left(\sin 2\theta \sin^2\theta\right)\, r^2}{2} \\
&= -1 - \frac{\sin 2\theta (1 - \cos 2\theta) r^2}{4} < 0, \quad r < 2
\end{aligned}$$

We see that for $r < 2$ the trajectories tend to the origin going (non-uniformly) as spirals clockwise around the origin.

This example demonstrates the main idea with applications of the LaSalles invariance principle to asymptotic stability of equilibrium points.

Proposition. Simple version of applying LaSalle's invariance principle for asymptotic stability of equilibrium points by using "weak" Lyapunof functions.

(The complete version of LaSalle's invariance principle is Theorem 5.15. p. 183 that is considered a bit later)

We find a simple "weak" Lyapunov function $V_f(z) \leq 0$ for $z \in U$ in the domain $U \subset G$, $0 \in U$. This fact implies stability of the equilibrium. Then we check what happens on the set $V_f^{-1}(0)$ where $V_f(z) = 0$. If the set $V_f^{-1}(0)$ contains no other orbits except the equilibrium point, this equilibrium point in the origin must be asymptotically stable.

Any trajectory starting in W will have a positive orbit with compact closure. We need this property for applying LaSalle's invariant principle describing ω- limit sets for positive orbits of solutions to ODEs.

Exercise.

Show that all trajectories of the system

$$
\begin{aligned}
x' &= y \\
y' &= -x - (1 - x^2)y
\end{aligned}
$$

that go through points in the domain $\left\| [x, y]^T \right\| < 1$, tend to the origin. Or by other words, show that the origin is an asymptotically stable equilibrium and that the circle $\left\| [x, y]^T \right\| < 1$ is it's domain of attraction.

Consider $V(x, y) = x^2 + y^2$.

$$
\begin{aligned}
V_f(x, y) &= 2xy - 2xy - (1 - x^2)y^2 = -(1 - x^2)y^2 \leq 0 \\
V_f^{-1}(0) &= \{(x, y) : y = 0\}
\end{aligned}
$$

The only invariant set is $\{0\}$, therefore for trajectories starting in $\left\| [x, y]^T \right\| < 1$ the origin is and attractor and it is asymptotically stable with $\left\| [x, y]^T \right\| < 1$ being the domain of attraction.

More general formulation and a proof of the LaSalle's invariance principle use some general properties of transition mappings, and ω - limit sets. We collect them here and give some comments about their proofs.

We consider I.V.P. and corresponding transition map $\varphi(t,\xi)$ for the system

$$\begin{aligned} x' &= f(x) \\ x(0) &= \xi \end{aligned}$$

with $f : G \to \mathbb{R}^n$, G - open, $G \subset \mathbb{R}^n$, f is locally Lipschitz, $\xi \in G$.

1.1 Main theorem on the properties of limit sets.

The next theorem on the properties of ω - limit sets collects properties of ω - limit sets valid for systems of any dimension, in contrast with the Poincare - Bendixson theorem and it's generalization, that gives a

description of ω - limit sets only for systems in plane, or on 2-dimensional manifolds.

Main theorem about properties of limit sets. Theorem 4.38, p.143

We keep the same limitations and notations for the autonomous system as above.

Let $\xi \in G$. Let the closure of the postive semi-orbit $O^+(\xi)$ be compact and contained in G,

Then $\mathbb{R}_+ \subset I_\xi$ and the ω - limit set $\Omega(\xi) \subset G$ is

1) non-empty

2) compact (bounded and closed)

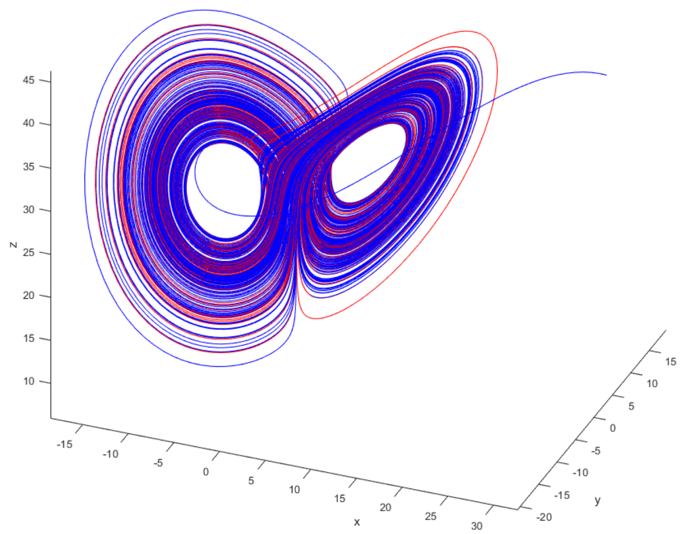
3) connected

4) invariant (both positively and negatively) under the local flow $\varphi(t,\xi)$ generated by the ODE: namely for any ω - limit point $\eta \in \Omega(\xi)$, the maximal interval $I_\eta = \mathbb{R}$ for initial data in η, and $\varphi(t,\eta) \in \Omega(\xi)$ for all $t \in \mathbb{R}$.

5) $\varphi(t,\xi)$ approaches $\Omega(\xi)$ as $t \to \infty$:

$$\lim_{t\to\infty} \operatorname{dist}(\varphi(t,\xi),\, \Omega(\xi)) = 0$$

Example. The Lorentz equation. Trajectory - blue, limit set $\Omega(\xi)$ - red



$$\begin{aligned} x' &= -\sigma(x-y) \\ y' &= rx - y - xz \\ z' &= xy - bz \end{aligned}$$

A trajectory for $\sigma = 10$, $r = 28$, $b = 8/7$.

Remark

The most interesting statement in the theorem is statement 4). It means that ω - limit sets consist of orbits of solutions to the system. Taking a starting point η on the limit set $\Omega(\xi)$ we get a trajectory $\varphi(t, \eta)$ that stays within this set $\Omega(\xi)$ infinitely long both in the future and in the past.

A simple tool to satisfy conditions in this theorem is to find a compact positively invariant set for the system that contains the point ξ. It can be done using one of two methods discussed earlier.

Proofs of statements in the Theorem **4.38,** are based on: general properties of compact sets for 1) ,2), simple contradiction arguments and the definition of limit sets for 3) and the translation property of the transition mapping $\varphi(t,\xi)$, together with continuity of $\varphi(t,\xi)$ for 4), and a contradiction argument togehter with the definition of ω - limit sets for 5).

We will give a **proof to 4).**

Let η be a limit point for $\varphi(t,\xi)$: $\eta \in \Omega(\xi)$. By definition there is a sequence of times $\{t_n\}$, $t \to \infty$ such that $\varphi(t_n,\xi) \to \eta$.

Consider the trajectory $\varphi(t,\eta)$ starting at η. Denote by I_η corresponding maximal interval and consider an arbitrary $t \in I_\eta$, belonging o the maximal interval I_η. We like to show that $\varphi(t,\eta) \in \Omega(\xi)$ that a trajectory starting in a limit point stays in the limit point forever in future and in the past.

For n large enough $t + t_n \overset{def}{=} s_n \in I_\xi$ - belongs to the maximal interval I_ξ of the solution $\varphi(t,\xi)$ for n large enough.

We apply the group relation for φ (similar to Chapmen-Kolmogorov relation for linear systems)

$$\varphi\left(s_n,\xi\right) = \varphi\left(t + t_n,\xi\right) = \varphi\left(t,\varphi(t_n,\xi)\right)$$

It is possible since the domain D of $\varphi(.,.)$ is open, $(t,\eta) \in D$ therefore there is a ball B around (t,η) such that $(t,\varphi(t_n,\xi)) \in B \subset D$ for n large enough because $\varphi(t_n,\xi) \to \eta$. Therefore $t \in I_{\varphi(t_n,\xi)}$.

By continuity of φ it follows:

$$\varphi\left(s_n,\xi\right) = \varphi\left(t + t_n,\xi\right) = \varphi\left(t, \overset{\lim = \eta}{\varphi(t_n,\xi)}\right) \to \varphi(t,\eta), \quad n \to \infty$$

It means that $\varphi(t,\eta)$ is a limit point for $\varphi\left(t,\xi\right)$ for any $t \in I_\eta$.■

LaSalle's invariance principle

We formulate now LaSalle's invariance principle that generalizes ideas that we discussed in the introductory example and gives a handy instrument for localizing ω - limit sets of non-linear systems in arbitrary dimension.

Theorem 5.12, p.180

Assume that f is locally Lipschitz $f : G \to \mathbb{R}^n$ as before and let $\varphi(t, \xi)$ denote the flow generated by the corresponding system

$$x' = f(x)$$

Let $U \subset G$ be non-empty and open. Let $V : U \to \mathbb{R}$ be continuously differentiable and such that $V_f(z) = \nabla V \cdot f(z) \leq 0$. for all $z \in U$. If $\xi \in U$ is such that the closure of the semi-orbit $O^+(\xi)$ is compact and is contained in U,

i) then $\mathbb{R}_+ \subset I_\xi$ (maximal existence interval for ξ) and

ii) as $t \to \infty$, $\varphi(t, \xi)$ approaches the largest invariant set contained in $V_f^{-1}(0)$ that is the set where $V_f(z) = 0$.

Proof.

Proof given in the solution of Exercise 5.9, on p. 312.

Set $x(t) = \varphi(t, \xi)$. By continuity of V and compactness of the closure $cl(O^+(\xi))$, V is bounded on $O^+(\xi)$ and therefore the function $V(\varphi(t, \xi))$ is bounded.

• Since

$$\frac{d}{dt}(V(x(t))) = V_f(x(t)) \leq 0$$

for all $t \in \mathbb{R}_+$, $V(x(t))$ is non-increasing. We conclude that the limit $\lim_{t\to\infty} V(x(t))$ of the non-increasing function $V(x(t))$ must exist and is finite. We denote it by λ:

$$\lim_{t\to\infty} V(x(t)) = \lambda$$

• Take an arbitrary an arbitrary point $z \in \Omega(\xi)$ in the ω - limit set $\Omega(\xi)$. Then by the definition of ω - limit sets, there is a sequence $\{t_n\}$ in $\mathbb{R}_+$

such that $\lim_{n\to\infty} t_n = \infty$ and

$$x(t_n) = \varphi(t_n, \xi) = x(t_n) \longrightarrow z, \quad n \to \infty$$

We apply V to the left and right hand side in this limit calulation.

For any continuous function F and any convergent sequence $\{g_n\}$it is valid that

$$F(\lim_{n\to\infty} g_n) = \lim_{n\to\infty} (F(g_n)$$

• By the continuity of V it follows that $V(z) = \lim_{n\to\infty} V(x(t_n))$ and $\lim_{n\to\infty} V(x(t_n)) = \lim_{t\to\infty} V(x(t))$. Therefore

$$V(z) = \lim_{n\to\infty} V(x(t_n)) = \lim_{t\to\infty} V(x(t)) = \lambda.$$

This key point in the proof $\Big(!!!\Big)$ implies that for all z in the omega limit set $\Omega(\xi)$ the test function V has the same value:

$$V(z) = \lambda, \quad \forall z \in \Omega(\xi) \tag{1}$$

• By the invariance of $\Omega(\xi)$ with respect to $\varphi(t, .)$, if $z \in \Omega(\xi)$, then $\varphi(t, z) \in \Omega(\xi)$ for all $t \in \mathbb{R}$.**(it is why the theorem is called the invariance ~principle!!!)**

Therefore $V(\varphi(t, z)) = \lambda$ for all $t \in \mathbb{R}$ is a constant function of time t. A constant function must have zero derivative:

$$\frac{d}{dt} V(\varphi(t, z)) = V_f(\varphi(t, z)) = 0$$

for all $t \in \mathbb{R}$. Since $\varphi(0, z) = z$ and z is an arbitrary point in $\Omega(\xi)$ it follows that

$$V_f(z) = \left.\frac{d}{dt} V(\varphi(t, z))\right|_{t=0} = 0, \quad \forall z \in \Omega(\xi) \tag{2}$$

and therefore $\Omega(\xi) \subset V_f^{-1}(0)$.

• The statement of the theorem follows now from the Main theorem about limit sets (Theorem 4.38), that states: $\Omega(\xi)$ is an invariant set under the action of $\varphi(t,.)$, and $\varphi(t,\xi)$ apporachs $\Omega(\xi)$ as $t \to \infty$.$\blacksquare$

Comment. It can be tempting to simplify the proof by concluding (1) from the fact that $(\nabla V)(z) = 0$ from all $z \in \Omega(\xi)$ which would imply (2).

However this conclusion is not valid, because the set $\Omega(\xi)$ is not open and therefore $V(z) = \lambda, \quad \forall z \in \Omega(\xi)$ does not imply $V_f(z) = 0, \quad \forall \in \Omega(\xi)$.

The invalidity of this conclusion is illustrated by the following simple example: $V(z) = \|z\|$, $\Omega(\xi) = \left\{z \in \mathbb{R}^N : \|z\| = 1\right\}$, then $v(z) = 1$ for all $z \in \Omega(\xi)$, but$(\nabla V)(z) = 2z \neq 0$ for all $z \in \Omega(\xi)$.

The following theorem follows rather directly from LaSalle's invariance principle and gives a practical criterium for asympototically stable equilibrium points using "weak" Lyapunov's functions.

Theorem 5.15. p. 183.

Let U be an open domain $U \subset G$, such that $0 \in U$ and a continuously differentiable function $V : U \to \mathbb{R}^n$ such that

$$V(0) = 0, \quad V(z) > 0, \forall z \in U \setminus \{0\}, \quad V_f(z) \leq 0, \forall z \in U \setminus \{0\}$$

and $\{0\}$ is the only invariant set contained in $V_f^{-1}(0)$, then 0 is an asymptotically stable equilibrium.$\square$

Proof follows from LaSalle's invariance principle and is a good exercise.

Theorem 5.22, p. 188. On global asymtotic stability

Assume that $G = \mathbb{R}^n$. Let the hypothesis of the Theorem 5.15 hold with $U = G = \mathbb{R}^n$.

Namely for a continuously differential function $V : \mathbb{R}^n \to \mathbb{R}$ such that $V(0) = 0$, $V(z) > 0$ for all $z \in U \setminus \{0\}$, $V_f(z) \leq 0$ for all $z \in U \setminus \{0\}$, the origin $\{0\}$ is the only invariant set contained in $V_f^{-1}(0)$.

If in addition the Lyapunov function V is radially unbounded:

$$V(z) \to \infty, \quad \|z\| \to \infty$$

then the origin 0 is a globally stable equilibrium that means that all solutions $\|\varphi(t,\xi)\| \to 0$, as $t \to \infty$.

Exercise 5.17

The aim of this exercise is to show that the condition of radial unboundedness in Theorem 5.22 is essential.

Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(z) = f(z_1, z_2) = \begin{cases} (-z_1, z_2) & \text{if } z_1^2 z_2^2 \geq 1 \\ (-z_1, 2z_1^2 z_2^3 - z_2) & \text{if } z_1^2 z_2^2 < 1 \,. \end{cases}$$

Define $V : \mathbb{R}^2 \to \mathbb{R}$ by

$$V(z) = V(z_1, z_2) = z_1^2 + \frac{z_2^2}{1+z_2^2} \,.$$

(a) Show that the equilibrium 0 of (5.1) is asymptotically stable.

(b) Show that the equilibrium 0 is *not* globally asymptotically stable.

(c) Show that V is not radially unbounded.

Examples of using La Salle's principle. Investigate stability of equilibrium points in the origin.

Example.

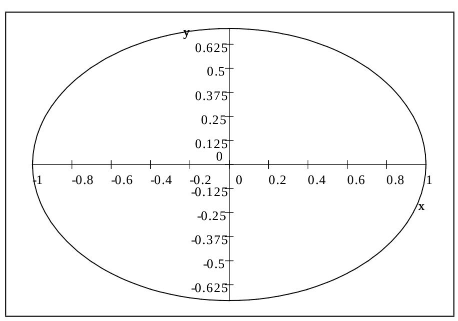
Consider the following system of ODEs: $\begin{cases} x' = & 2y \\ y' = & -x - (1 - x^2)y \end{cases}$.

Show the asymptotic stability of the equilibrium point in the origin and find it's domain of attraction. **(4p)**

Solution.

We try the test function $V(x, y) = x^2 + 2y^2$ that leads to cancellation of mixed terms in the directional derivative V_f along trajectories:

$V_f(x, y) = 4xy - 4xy - 4y^2(1 - x^2) = -4y^2(1 - x^2)$ that is not positive for $|x| \leq 1$. Therefore the origin is a stable stationary point. Checking the behavior of the system on the set of zeroes to $V_f(x, y)$ inside the stripe $|x| < 1$ we consider $(V_f)^{-1}(0) = \{(x, y) : y = 0, |x| < 1\}$. On this set $y' = -x$ and the only invariant set in $(V_f)^{-1}(0)$ is the origin. LaSalle's invariance principle implies that the origin is asymptotically stable and the domain of attraction is the largest set bounded by a level set of $V(x, y) = x^2 + 2y^2$ inside the stripe $|x| \leq 1$. The largest such set will be the interior of the ellipse $x^2 + 2y^2 = C$ such that is touches the lines $x = \pm 1$. Taking points $(\pm 1, 0)$ we conclude that $1 = C$. and the boundary of the domain of attraction is the ellipse $x^2 + 2y^2 = 1$ with halfs of axes 1 and $\sqrt{0.5}$:



The next theorem gives a simple criterion for having the whole space as the domain of attraction for an asymptotically stable equilibrium point.

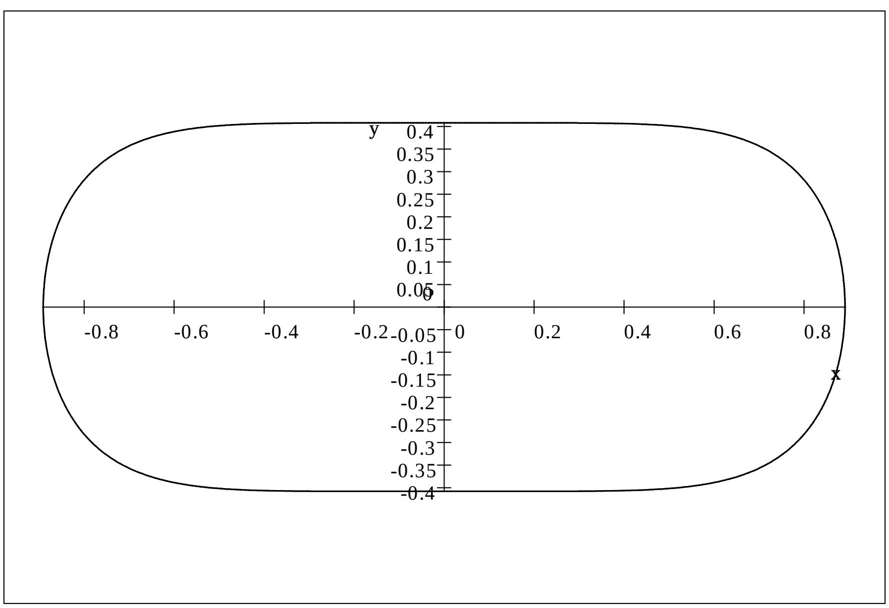
Example. Investigate stability of the equilibrium point in the origin.

$$
\begin{aligned}
x' &= -y - x^3 \\
y' &= x^5
\end{aligned}
$$

We try our simplest choice of the Lyapunov function: $V(x,y) = x^2 + y^2$ and arrive to

$$V_f(x,y) = -2xy - 2x^4 + 2yx^5$$

It does not work because the expression $V_f(x,y)$ includes two indefinite terms: $2xy$ and $2yx^5$ that change sign around the origin. We try a more flexible expression by looking on particular expressions in the right hand side of the equation: $V(x,y) = x^6 + \alpha y^2$ where $\partial V/\partial x = 6x^5$ with the same power of x as in the equation, and the parameter α that can be adjusted later. V is a positive definite function: $V(0) = 0$ and $V(z) > 0$ for $z \neq 0$.The level sets to V look as flattened in y - direction ellipses. The curve $x^6 + 3y^2 = 0.5$ is depicted:



$$V_f(x,y) = 6x^5(-y - x^3) + 2\alpha y x^5 = -6x^5 y + 2\alpha x^5 y - 6x^8$$

We get again two indefinite terms, but they are proportional and the choice $\alpha = 3$ cancels them:

$$V_f(x, y) = -6x^8 \leq 0$$

Therefore the origin is a stable equilibrium point. $V_f(x, y) = 0$ on the whole $y-$axis that in our "general" theory is denoted by $V_f^{-1}(0)$.We check invariant sets of the system on the set $V_f^{-1}(0)$. We observe that $x' = -x^3$ (only this fact is important) and $y' = 0$ (it does not matter for $V_f^{-1}(0)$ that is $y-$axis). Therefore $\{0\}$ is the only invariant set on the y - axis. Trajectories starting on the y - axis go across it in all points except $\{0\}$. The LaSalle's invariance principle implies that all trajectories approach $\{0\}$ as t tends to infinity and the origin is asymptotically stable.

The test function $V(z) \to \infty$ as $\|z\| \to \infty$. It implies that the whole plain is a region or domain of attraction for the equilibrium point in the origin.■