Main ideas and tools in the course in ODE

1. Integral form of I.V.P. to ODEs
2. Grönwall's inequality for showing uniqueness and continuity with respect to data.
3. Generalised eigenspaces of matrices. Basis of generalized eigenvectors.
4. Jordan form of matrices. Matrix exponent and logarithm.
5. Transfer matrix. Monodromy matrix.
6. Stability and instability of equilibrium points.
7. Linearization and Grobman Hartman theorem. (iff $\mathrm{Re}(\lambda) \neq 0$)
8. Lyapunov functions.(for stability and for hunting positively invariant sets)
9. LaSalle's invariance principle for hunting ω - limit sets.
10. Idea of solving integral equations by iterations.

Examples

1) Solve the initial value problem

$$\dot{x} = t\,x^3, \qquad x\,(1) = \xi$$

and find maximal intervals for solutions. Give a sketch of the domain for $x(t) = \varphi(t, 1, \xi)$ in the (t, x) plane.

2) Can one conclude which maximal interval have solutions to the similar equation

$$\dot{x} = t^3 x$$

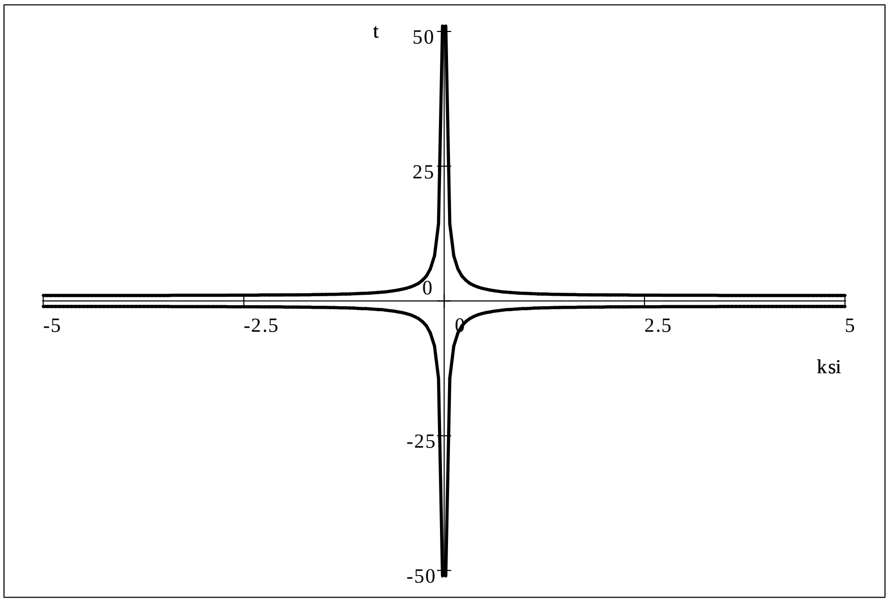
without solving it?

1. **Solution.**

1) It is the equation with separable variables.

$$\begin{aligned}
\frac{dx}{dt} &= tx^3; \qquad x(1) = \xi \\
\int \frac{dx}{x^3} &= \int tdt \\
\frac{-1}{2x^2} &= \frac{t^2}{2} - C \\
C &= \frac{t^2}{2} + \frac{1}{2x^2}; \quad C = \frac{1}{2} + \frac{1}{2\xi^2} = \frac{1+\xi^2}{2\xi^2} \\
\frac{-1}{2x^2} &= \frac{t^2}{2} - \frac{1+\xi^2}{2\xi^2} \\
\frac{-1}{2x^2} &= \frac{\xi^2 t^2}{2\xi^2} - \frac{1+\xi^2}{2\xi^2} = \frac{\xi^2 t^2 - \left(1+\xi^2\right)}{2\xi^2} \\
x^2 &= \frac{\xi^2}{\left(1+\xi^2\right) - \xi^2 t^2} = \frac{1}{\left(1+\xi^2\right)/\left(\xi^2\right) - t^2} \\
x &= \sqrt{\frac{1}{\left(1+\xi^2\right)/\left(\xi^2\right) - t^2}}, \left(1+\xi^2\right)/\left(\xi^2\right) - t^2 > 0, \xi > 0 \\
x &= -\sqrt{\frac{1}{\left(1+\xi^2\right)/\left(\xi^2\right) - t^2}}, \left(1+\xi^2\right)/\left(\xi^2\right) - t^2 > 0, \xi < 0 \\
x &\equiv 0, \quad \xi = 0, \quad \textit{equilibrium} \\
\left(1+\xi^2\right)/\left(\xi^2\right) &> t^2; \quad t \in \left(-\sqrt{\left(1+\xi^2\right)/\left(\xi^2\right)}, \sqrt{\left(1+\xi^2\right)/\left(\xi^2\right)}\right) \mathit{OPEN}!!!
\end{aligned}$$

1. The maximal intervals for these solutions are open in accordance with the general theory. One solution $x \equiv 0$ is defined on the whole $\mathbb{R}$. We draw boundaries of the domain for $\varphi(t, 1, \xi)$.



The equation $\dot{x} = t^3 x$ is defined on $\mathbb{R} \times \mathbb{R}$ and the right hand side satisfies on any compact time interval $[-R, R]$, $R > 0$ the estimate $\left|t^3 x\right| \le R^3(1 + |x|)$ where the right hand side rises linearly with respect to $|x|$. It implies that the maximal existence interval for all solutions to this equation is $\mathbb{R}$.

Estimating Lyapunov functions V and their derivatives $V_f = \nabla V \cdot f$ along trajectories.

Investigation of positivity of functions V and $V_f = \nabla V \cdot f$.

Choosing a Lyapunov function: it must be positive definite: $V(0) = 0$, $V(x) > 0$, $x \neq 0$.

We like to have $V_f = \nabla V \cdot f$ negative definite $V_f < 0$ or at least $\nabla V \cdot f \le 0$.

Example.

1. Consider the following system of ODE: $\left\{ \begin{array}{ll} x' = & -x - 2y + xy^2 \\ y' = & 3x - 3y + y^3 \end{array} \right.$.

   Show asymptotic stability of the equilibrium point in the origin and find the region of attraction for that.

   Hint: applying Lyapunovs theorem, use the elementary inequality

   $$|xy| \le \frac{1}{2}\left(x^2 + y^2\right)$$

   to estimate indefinite terms with xy.

A more general inequality can be useful for polynomials of higher degree in f:

$$|ab| \leq \frac{a^p}{p} + \frac{b^q}{q}; \qquad \frac{1}{p} + \frac{1}{q} = 1, \quad p, q > 1$$

1. **Solution.** Choose a test function $V(x, y) = \frac{1}{2}\left(x^2 + y^2\right)$

   $V_f = x(-x - 2y + xy^2) + y\left(3x - 3y + y^3\right) = xy - x^2 - 3y^2 + y^4 + x^2y^2$
   $= -x^2\left(1 - y^2\right) - y^2\left(3 - y^2\right) + xy \leq -x^2\left(1 - y^2\right) - y^2\left(3 - y^2\right) + 0.5x^2 + 0.5y^2$

   We apply the inequality $2xy \leq \left(x^2 + y^2\right)$ to the last term and collecting terms with x^2 and y^2 arrive to the estimate

   $V_f \leq -x^2\left(0.5 - y^2\right) - y^2\left(2.5 - y^2\right)$

   It implies that $V_f < 0$ for $(x, y) \neq (0, 0)$ and $|y| < 1/\sqrt{2}$.Therefore the origin is asymptotically stable.

   The attracting region is bounded by the largest levle set of V - a circle having the center in the origin that fits to the domain $|y| < 1/\sqrt{2}$, namely $\left(x^2 + y^2\right) < 1/2$.

   The second idea for choosing Lyapunov functions is choice of V of **polynomilas with arbitrary even powers and arbitrary coefficients.**

   Another more clever choice of a test function as

   $$V(x, y) = ax^m + by^n$$

   in particular $V(x, y) = 3x^2 + 2y^2$ works in this particular case:

   $V_f = 6x(-x-2y+xy^2)+4y(3x-3y+y^3) = 4y^4-12y^2-6x^2+6x^2y^2 = -4y^2\left(3 - y^2\right) - 6x^2\left(1 - y^2\right) < 0$

   for $|y| < 1$, therefore the ellipse $3x^2 + 2y^2 < 2$ is a domain of attraction for the asymptotically stable equilibrium in the origin.

   One can also observe the asymptotic stability of the origin here by linearization with variational matrix

   $A = \begin{bmatrix} -1 & -2 \\ 3 & -3 \end{bmatrix}$, with characteristic polynomial: $\lambda^2 + 4\lambda + 9 = 0$, and calculating eigenvalues: $-i\sqrt{5}-2, i\sqrt{5}-2$ with $\operatorname{Re}\lambda < 0$. But linearization gives no information about the domain of attraction.

Poincare - Bendixson theorem and testing absence of equilibrium points in the positive invariant set.

We try to find an ring shaped domain that is positively invariant and need to check three conditions:

i) The outer boundary of the ring (using a level set of a test function, or a polygon shaped domain testing velosities on each segment of it's boundary)

ii) The inner boundary of the ring (using a level set of a test function, or linearization for identifying a repeller inside a large postively invariant set by applying the Grobman - Hartman theorem)

iii) Check that no equilibrium points exist inside of the ring (missed often by students)

Example. Show that the following system of ODEs has a periodic solution.

$$\begin{cases} x' = x - 2y - x\left(2x^2 + y^2\right) \\ y' = 4x + y - y\left(2x^2 + y^2\right) \end{cases} \qquad \textbf{(4p)}$$

Solution. Consider the following test function: $V(x, y) = 2x^2 + y^2$. Denoting the right hand side in the equation by vectorfunction $F(x, y)$ we conclude that

$V_f = \nabla V \cdot f = 4x^2 - 8xy - 4x^2\left(2x^2 + y^2\right) + 8xy + 2y^2 - 2y^2(2x^2 + y) = 2\left(1 - (2x^2 + y^2)\right)(2x^2 + y^2)$.

It implies that the elliptic shaped ring: $R = \left\{(x, y) : 0.5 \leq (2x^2 + y) \leq 2\right\}$ is a positive invariant compact set for the ODE, because velocity vectors are directed inside of this ring both on it's outer and inner boundaries ($\nabla V \cdot F < 0$ for $(2x^2 + y) = 2$ and $\nabla V \cdot F > 0$ for $(2x^2 + y) = 0.5$.

The origin is the only equilibrium point of the system. It is not so easy to see from the system of equations itself. But one can see it easier by cheching first zeroes of $V_f(x, y)$ that is a scalar function and evidently must be zero in all equilibrium points..

We observe that $V(x, y) = 2x^2 + y^2$ is positive definite and $\nabla V \cdot F(x, y) = 0$ only if $(x, y) = (0, 0)$ or if $(2x^2 + y^2) = 1$.But it is easy to see from the expression for the right hand side for the ODE that in the last case (x, y) cannot be equilibrium point because the right hand side becomes linear with nondegenerate matrix and is zero only in the origin $(x, y) = (0, 0)$. The equation for equilibrium points on the level set $(2x^2 + y^2) = 1$ is the following:

1. $$\begin{cases} 0 = x - 2y - x = -2y \\ 0 = 4x + y - y = 4x \end{cases}$$

   By the Poincare-Bendixson theorem the positively invariant set R not including any equilibrium point must include at least one orbit of a periodic solution.$\blacksquare$