MVE165, Applied Optimization
Lecture 1

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Course contents

- Applications of optimization
- Mathematical modelling
- Solution techniques
- Solvers

Model types

- Linear optimization
- Linear network models
- Integer linear optimization
- Unconstrained and constrained non-linear optimization
- Optimization under uncertainty
- Constraint programming

Organization

- Lectures – mathematical optimization modelling and theory
- Exercises – use solvers, oral examination
- Guest lectures – applications of optimization
- Assignments – modelling, use solvers, written reports, opposition, & oral presentations

Staff

- Ann-Brith Strömberg, course leader and lecturer
- Birgit Grohe, lecturer
- Michael Patriksson, examiner
- Fredrik Hedénus (Physical Resource Theory), guest lecturer
- Mats Viberg (Signals and Systems), guest lecturer
- Caroline Olsson (Radiation Physics), guest lecturer

Course homepage

`http://www.math.chalmers.se/Math/Grundutb/CTH/mve165/0708/`

- Contains details, information on assignments and exercises, deadlines, lecture notes, etc
- Will be updated **every** week during the course
- The course information (ps & pdf) was updated 080331 !!

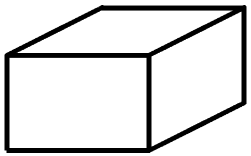
Optimization

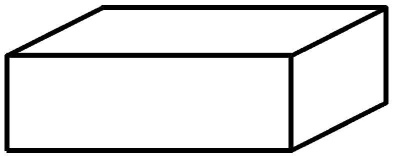
"Do something as good as possible"

- Something: Which are the decision alternatives?
- Possible: What restrictions are there?
- Good: What is a relevant optimization criterion?

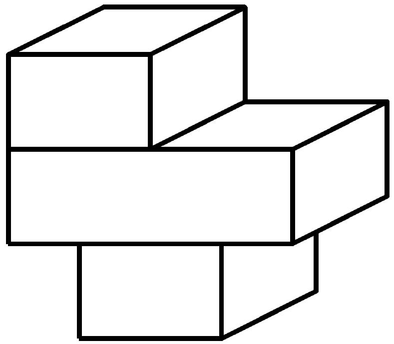
A manufacturing example: Produce tables and chairs from "small" and "large" pieces

Small piece

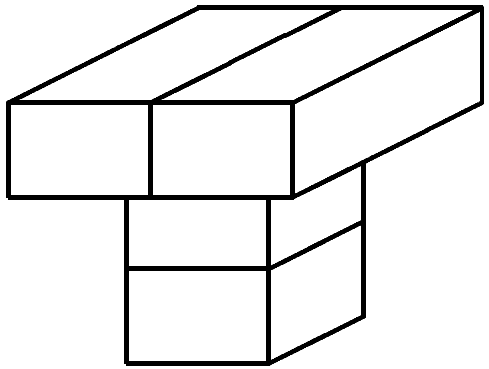




Large piece



Chair



Table

A manufacturing example, continued

- A chair is assembled from one large and two small pieces
- A table is assembled from two pieces of each
- Only 6 large and 8 small pieces are available
- A table/chair is sold for 1600:-/1000:-
- Assume that all items produced can be sold and determine an optimal production plan.

A mathematical optimization model

Something: Which are the decision alternatives? $\Rightarrow$ *Variables*

$$x_1 = \text{number of tables produced and sold}$$

$$x_2 = \text{number of chairs produced and sold}$$

Possible: What restrictions are there? $\Rightarrow$ *Constraints*

$$\begin{array}{rcrcll}
2x_1 & + & x_2 & \leq & 6 & \text{(6 large pieces)} \\
2x_1 & + & 2x_2 & \leq & 8 & \text{(8 small pieces)} \\
 & & x_1,\ x_2 & \geq & 0 & \text{(physical restrictions)}
\end{array}$$

Good: What is a relevant optimization criterion? $\Rightarrow$ *Objective function*

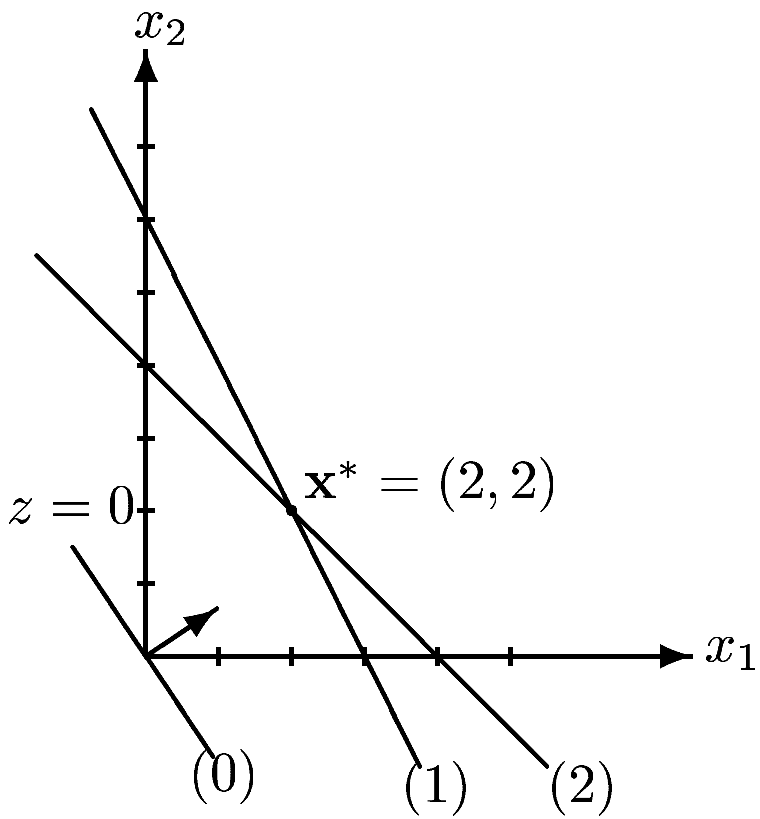
$$\text{maximize} \quad z = 1600x_1 + 1000x_2 \qquad (z = \text{total revenue})$$

Solve the model using LEGO!

- Start at no production: $x_1 = x_2 = 0$. Use the "best marginal profit" to choose the item to produce.
  - x_1 has the highest marginal profit (1600:-/table) $\Rightarrow$ produce as many tables as possible.
  - At $x_1 = 3$, there are no more large pieces left.
- The marginal value of x_2 is now 200:- since taking apart one table (-1600:-) yields two chairs ($+2000$:-) $\Rightarrow$ 400:-/2 chairs.
  - Increase x_2 maximally $\Rightarrow$ decrease x_1.
  - At $x_1 = x_2 = 2$ there are no more small pieces.
- The marginal value of x_1 is negative (to build one more table one has to take apart two chairs $\Rightarrow$ -400:-). The marginal value of x_2 is -600:- (to build one more chair one table must be taken apart). Hence $x_1 = x_2 = 2$ is an optimal solution.

Geometrical representation of the model

$$\begin{array}{lrcrcrcl} \text{maximize} & z & = & 1600x_1 & + & 1000x_2 & & \\ \text{subject to} & & & 2x_1 & + & x_2 & \leq & 6 \\ & & & 2x_1 & + & 2x_2 & \leq & 8 \\ & & & & & x_1,\ x_2 & \geq & 0 \end{array}$$



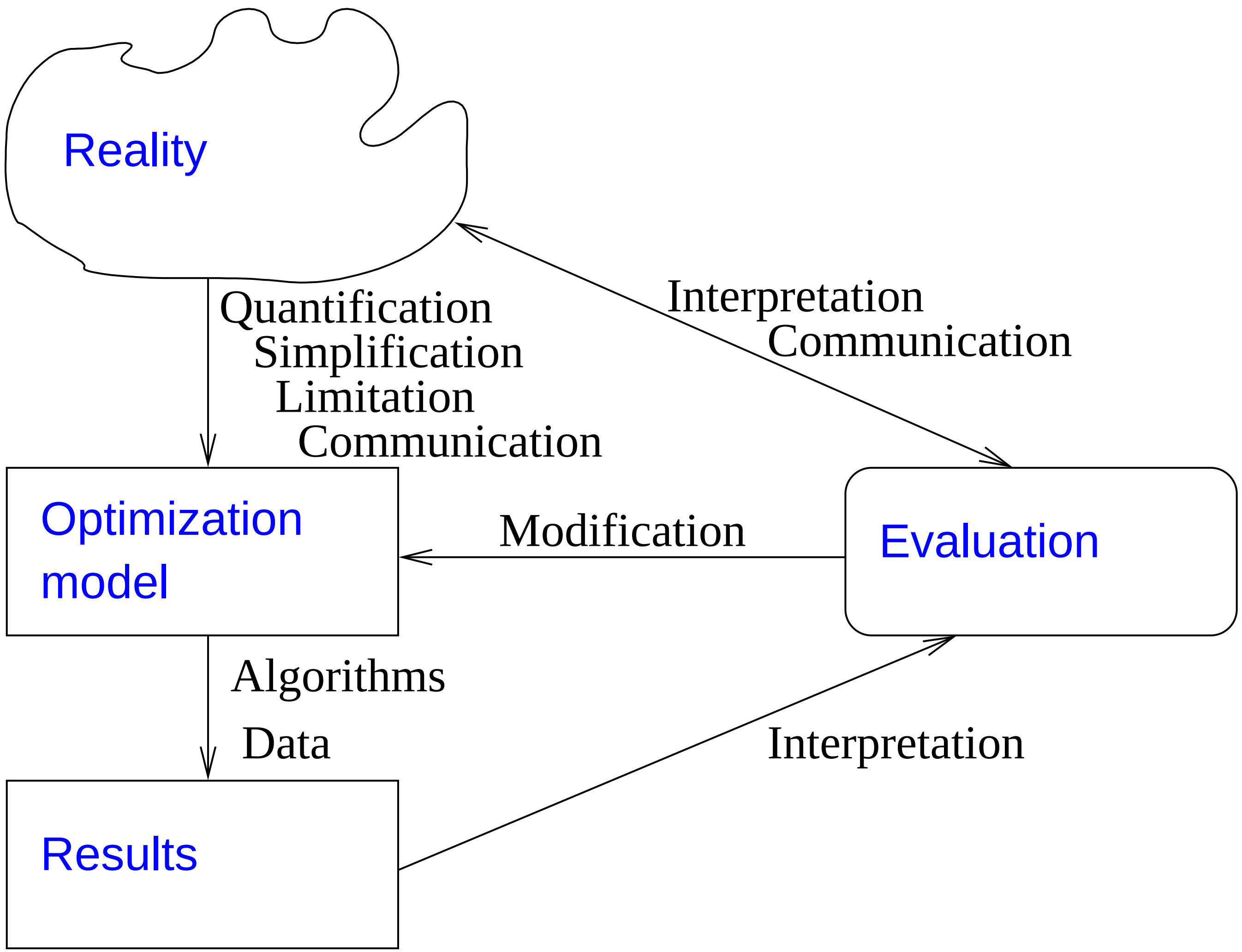
The optimal solution happens to be integral

Why?

Is this always the case?

E.g. 7 small pieces ...

Operations research—more than just mathematics



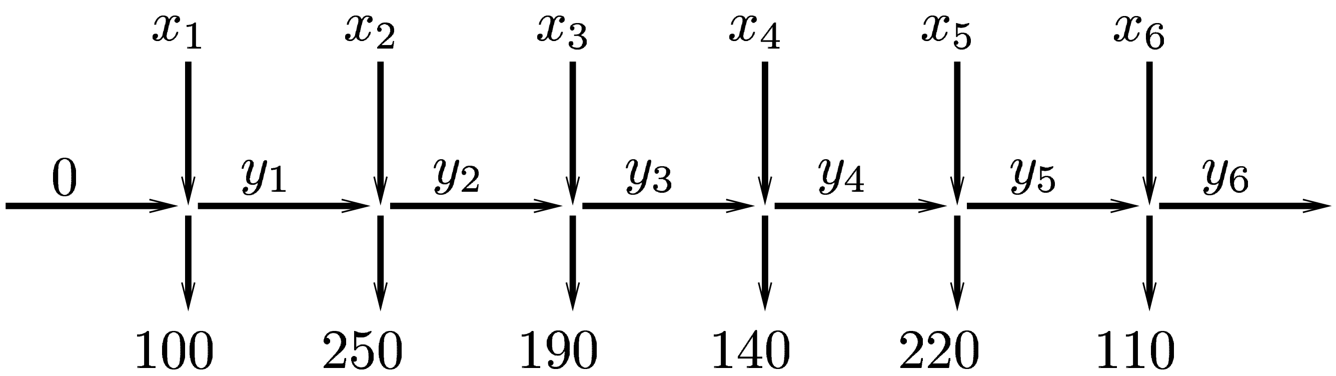
Modeling—a production-inventory example

- Deliver windows over a six-month period
- Demand for each month: 100, 250, 190, 140, 220, and 110 units
- Production cost/window: €50,€45, €55, €48, €52, and €50
- Storing a produced window from one month to the next costs €8
- Meet the demands and minimize costs
- Find an optimal production schedule

Define the decision variables

$$
\begin{aligned}
x_i &= \text{number of units produced in month } i = 1, \ldots, 6 \\
y_i &= \text{units left in the inventory at the end of month } i = 1, \ldots, 6
\end{aligned}
$$

The “flow” of windows over time can be illustrated as:



Define the limitations/constraints

- For each month:

  initial inventory + production − ending inventory = demand

$$
\begin{array}{rcrcrcrl}
0 & + & x_1 & - & y_1 & = & 100, & \\
y_1 & + & x_2 & - & y_2 & = & 250, & \\
y_2 & + & x_3 & - & y_3 & = & 190, & \\
y_3 & + & x_4 & - & y_4 & = & 140, & \\
y_4 & + & x_5 & - & y_5 & = & 220, & \\
y_5 & + & x_6 & - & y_6 & = & 110, & \\
 & & x_i & , & y_i & \geq & 0, & i = 1, \ldots, 6
\end{array}
$$

Objective function: minimize the costs

- Production cost (€):

$$50\ x_1 + 45\ x_2 + 55\ x_3 + 48\ x_4 + 52\ x_5 + 50\ x_6$$

- Inventory cost (€):

$$8\ (y_1 + y_2 + y_3 + y_4 + y_5 + y_6)$$

$\Rightarrow$ Objective function (€):

$$\begin{aligned} \text{minimize} \quad & 50x_1 + 45x_2 + 55x_3 + 48x_4 + 52x_5 + 50x_6 \\ & +8(y_1 + y_2 + y_3 + y_4 + y_5 + y_6) \end{aligned}$$

A complete and general optimization model

$$
\begin{array}{lrcll}
\text{minimize} & \displaystyle\sum_{i=1}^{6} c_i x_i + \sum_{i=1}^{6} k y_i, & & & \\
\text{subject to} & y_{i-1} + x_i - y_i & = & d_i, & i = 1, \ldots, 6, \\
& y_0 & = & 0, & \\
& x_i, y_i & \geq & 0, & i = 1, \ldots, 6,
\end{array}
$$

where the demands are given by the vector

$$d = (d_i)_{i=1}^{6} = (100, 250, 190, 140, 220, 110),$$

the production costs per item are given by the vector

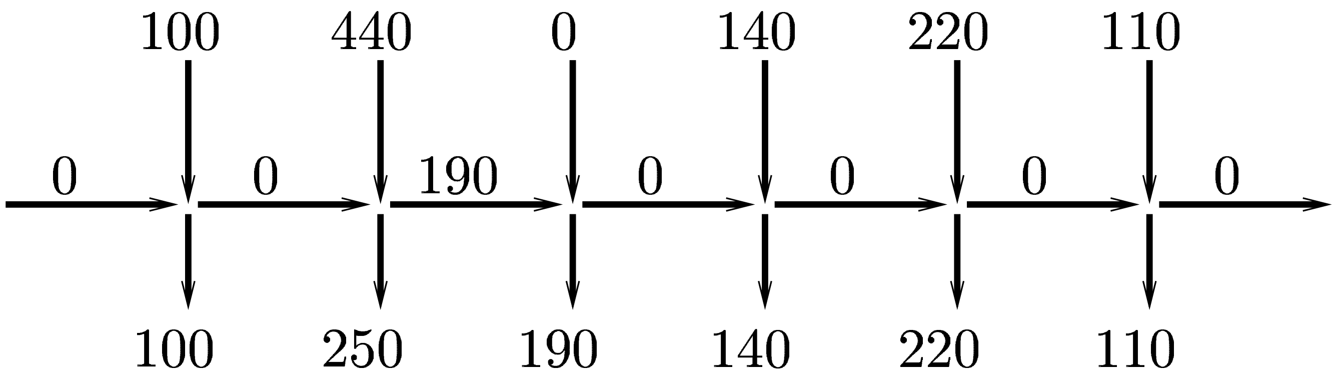
$$c = (c_i)_{i=1}^{6} = €\,(50, 45, 55, 48, 52, 50),$$

and the inventory cost each month is given by $k =$€ 8 per item.

An optimal solution—optimal production schedule

$$x = (x_i)_{i=1}^{6} = (100, 440, 0, 140, 220, 110)$$

$$y = (y_i)_{i=0}^{6} = (0, 0, 190, 0, 0, 0, 0)$$



- The minimal total cost is €49980
- Homework: What would be the solution if the production capacity was 220 items per month? If the inventory cost was reduced to €2/month and item?

Maximize the profit from a family dairy[a]

- Three cows produce together 85l of milk each week
- Turn the milk into ice cream and butter, sell on Saturday market
- 1kg of butter requires 8l of milk (+salt—simplification)
- 1l of ice cream requires 3l of milk (+egg, sugar, vanilla—simplif.)
- The freezer can hold at most 23l of ice cream
- The available hours of work per week is 6
- 1h work needed to produce either 15l ice cream or 1kg butter
- Any fraction of work time yields the corresponding fraction of product (simplification)
- The products have good reputation—everything is sold (simplif.)
- Prices ensure a profit of €2/l of ice cream and €3/kg of butter
- How much should be produced of each product to maximize the total profit?

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[a] Adapted from Ferris et al.: "Linear programming with MATLAB" (2007)

Family dairy: Modeling

- Identify decision variables: Which quantities can be varied?

⇒ Amount of ice cream (liters) and butter (kilograms) to produce:

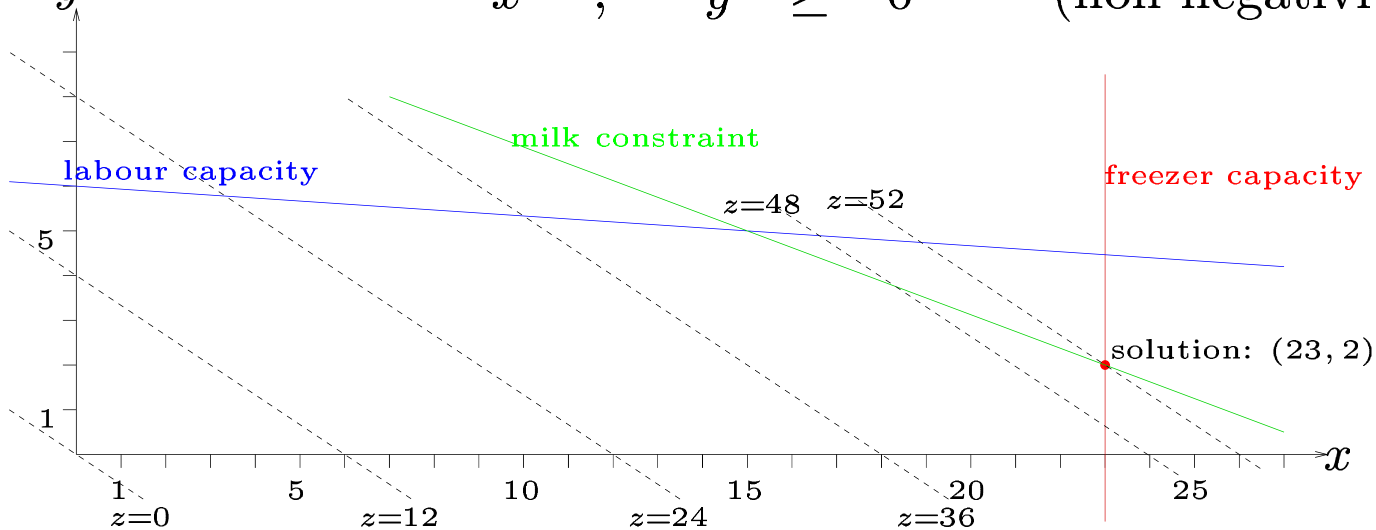
$$x = \text{number of liters of ice cream per week}$$
$$y = \text{number of kilograms of butter per week}$$

⇒ Objective function, resulting total profit (€): $z = 2x + 3y$

- Formulate constraints to prevent from infeasible solutions:
  - Freezer capacity (liters): $x \leq 23$
  - Work time (hours): $\frac{1}{15}x + y \leq 6$
  - Milk needed for production (liters of milk): $3x + 8y \leq 85$
- No negative production: $x \geq 0, y \geq 0$

Family dairy: Linear program & graphical solution

$$
\begin{array}{llrcrcl l}
\text{maximize} & z = & 2x & + & 3y & & & \text{(profit)} \\
\text{subject to} & & x & & & \leq & 23 & \text{(freezer capacity)} \\
 & & 0.067x & + & y & \leq & 6 & \text{(labour capacity)} \\
 & & 3x & + & 8y & \leq & 85 & \text{(milk constraint)} \\
 & & x & , & y & \geq & 0 & \text{(non-negativity)}
\end{array}
$$



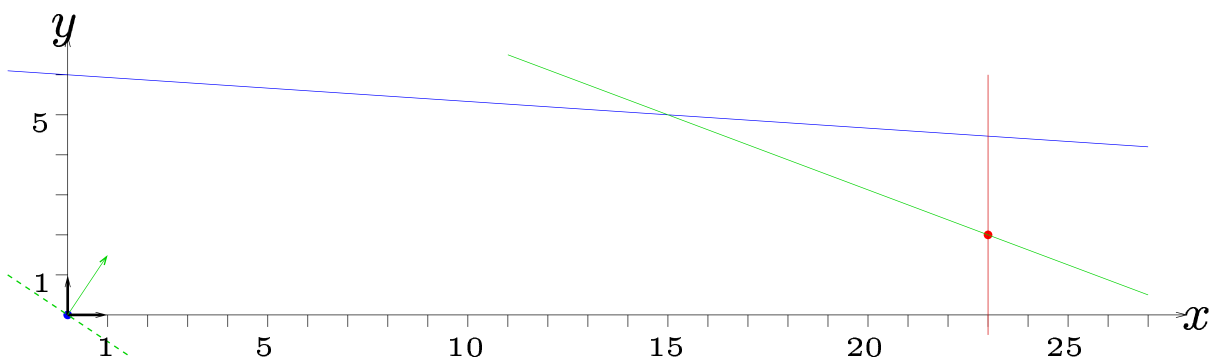
- Maximal profit: $z =$ €52 at the extreme point $(x, y) = (23, 2)$
- Optimal solution: 23l ice cream and 2kg butter per week

Solution method in graphical terms

- Objective function and constraints are *linear*

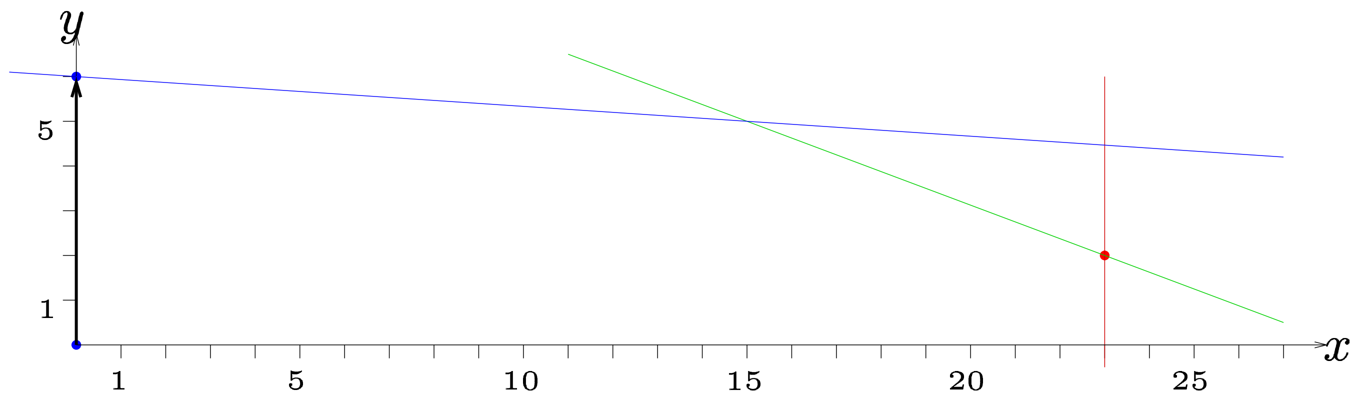
⇒ An optimal solution can *always* be found at an *extreme point* to the feasible set. Why?

- The *simplex method*: search for an optimal solution among the extreme points to the feasible set—in a structured way
  - Start at the extreme point $(x, y) = (0, 0)$ where $z = 0$



  - Increasing y yields $z = 3y$; increasing x yields $z = 2x$

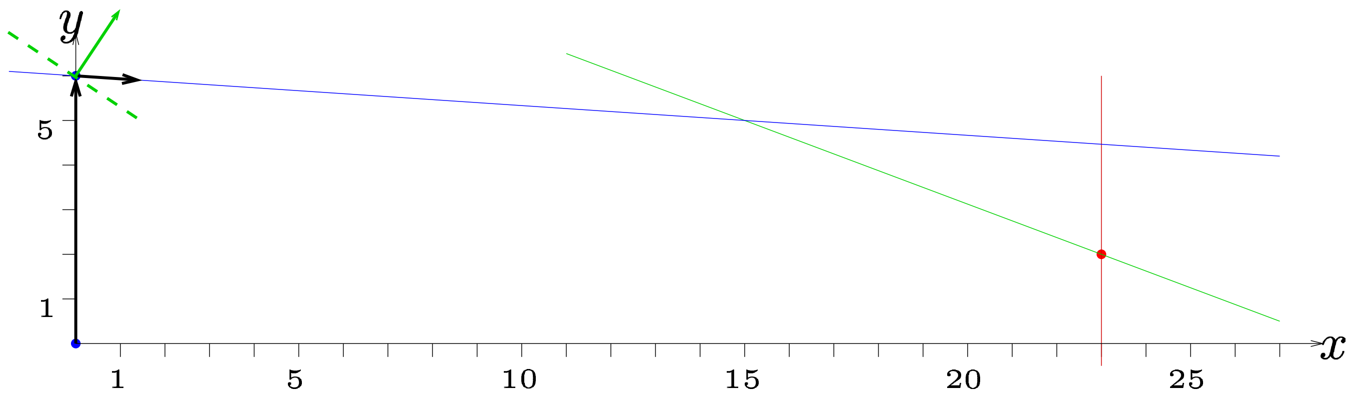
⇒ Increase y to get the fastest increase in z

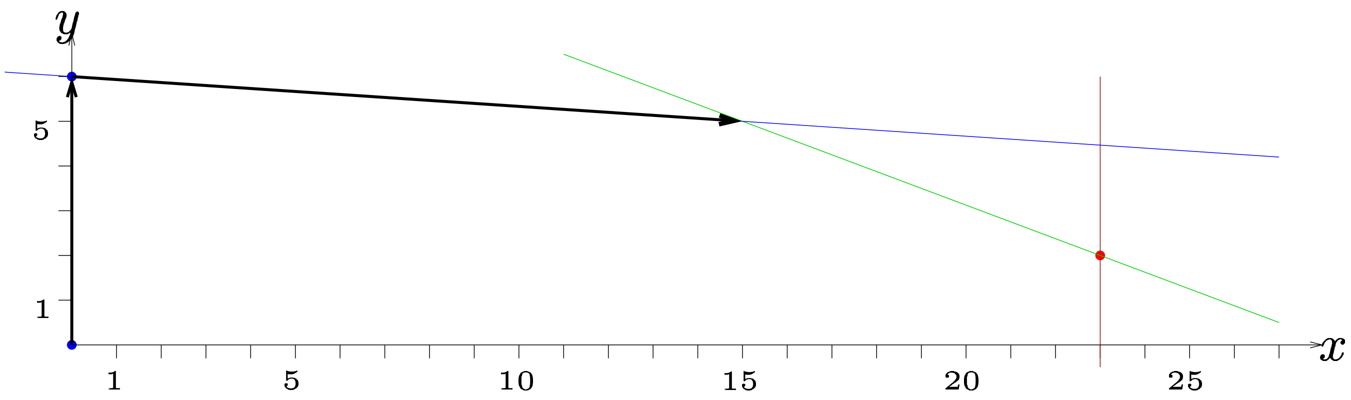


– At $(x, y) = (0, 6)$ $(z = 18)$ the *labour constraint* gets *binding*

– Increasing x from 0 increases z and forces a decrease of y due to the *labour constraint*: $y = 6 - 0.067\Delta x$ (where $\Delta x > 0$)

$$z = 2\Delta x + 3(6 - 0.067\Delta x) = 18 + 1.8\Delta x > 18$$

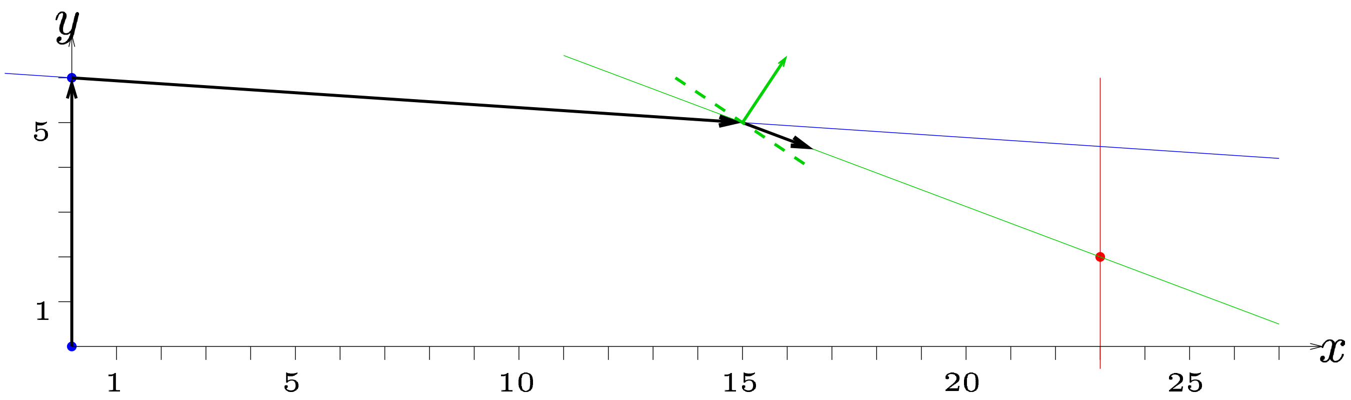


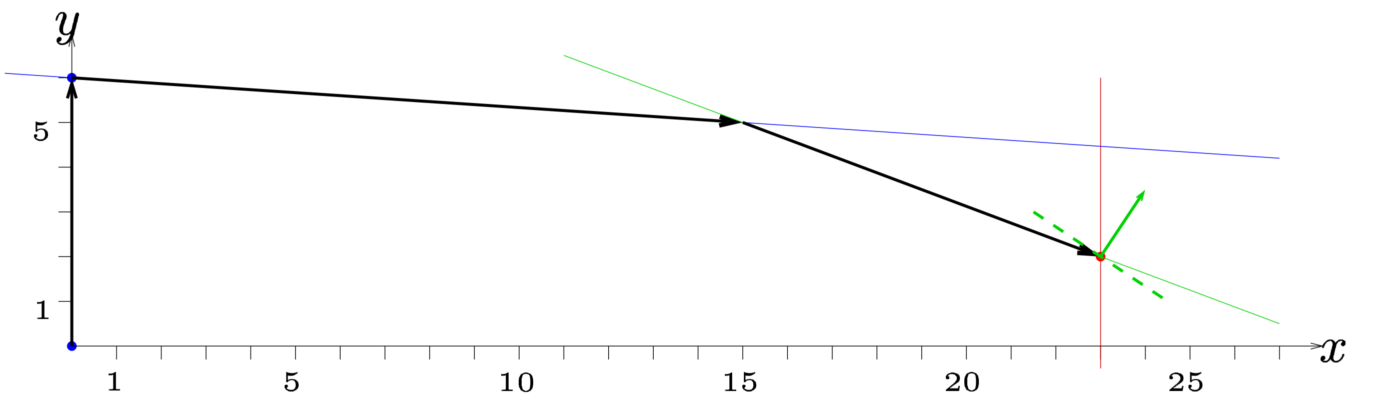


– At $(x, y) = (15, 5)$ $(z = 45)$ the *milk constraint* gets *binding*

– Increasing x from 15 further increases z and forces y to decrease due to the *milk constraint*: $y = \frac{85-3(15+\Delta x)}{8} = 5 - \frac{3}{8}\Delta x$

$$z = 2(15 + \Delta x) + 3(5 - \tfrac{3}{8}\Delta x) = 45 + 0.875\Delta x > 45 \quad (\Delta x > 0)$$





- At $(x, y) = (23, 2)$ $(z = 52)$ the *freezer constraint* gets *binding*
- x cannot be increased from 23
- Decreasing y yields $z = 2 \cdot 23 + 3 \cdot (2 - \Delta y) = 52 - 3\Delta y < 52$
- Decreasing x yields $z = 2 \cdot (23 - \Delta x) + 3 \cdot 2 = 52 - 2\Delta x < 52$
- No further improvements can be made, no extreme point can be better

- The optimal solution is given by $x = 23$ and $y = 2$ with optimal value $z = 52$

- The freezer and milk constraints are binding at the optimal point

The simplex method for linear programs

- An optimal solution can always be found at an extreme point of the feasible set
- Several extreme points can be optimal. Why? When?
- Utilizes linear algebra and the constraint inequalities/equations to detect extreme point solutions
- The objective function "guides" a path through the extreme points with increasing/decreasing values (max/min)
- Stops when no further improvements can be made
- Solves general linear programs with n variables and m constraints
- The number of extreme points is $\leq \frac{n!}{m!(n-m)!}$
- Typically, the number of extreme points visited by the simplex method is $\leq 3m$

Linear programming solvers

- MATLAB optimization toolbox
- CPLEX (commercial software, free student versions, interface to AMPL and MATLAB)
- GLPK (free software, interface to MATLAB)
- Clp (interface to MATLAB)
- Excel Solver (in the Taha book but not in this course)
- TORA (in the Taha book but not in this course)

Modelling software

- AMPL (A Mathematical Programming Language)

Recommended exercises[a]

- Problem set 2.1A 1–4
- Problem set 2.2A 1–18
- Problem set 2.2B 1–7
- Problem set 2.3A 3
- Problem set 2.3D 2, 4, 6
- Problem set 2.3E 1–5
- Problem set 2.3F 1–6
- Problem set 2.3G 1–2, 12

[a] Do at least one or two exercises from each problem set