Chalmers University of Technology

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A Short Introduction to Constraint Programming

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Constraint Satisfaction Problem (CSP)

- a set of **variables** $X = \{x_1, \ldots, x_n\}$,
- a discrete, finite **domain** for each variable $dom(x_i) = \{v_1, \ldots, v_d\}$,
- a set of **constraints** over subsets of variables.

Solution to a CSP: assignment of values to each variable such that all constraints are fulfilled.

Constraints

- $x_1 + 2x_2 \geq 4$ (linear constraint)
- $sin(x_1) = log(x_2)$ (nonlinear constraint)
- $x_1 \vee x_2 \vee x_3$ (logical constraint)
- all_diff, resource constraints, etc. (*global* constraints)
- general constraints (e.g a truth table)

Problems that can be modelled as a CSP

- graph coloring problems
- cryptarithmetic puzzles
- n-queens problem
- Satisfiability problem (SAT)
- Frequency Assignment problem (FAP)
- optimization problems → *Constraint Optimization Problems (COP)*
- ...

There always exist several ways of modelling a problem!

A Graph Coloring Problem

Color the map of Australia with three colors such that no two adjacent areas (territories) get the same color.

Picture: Australian map and constraint graph from 'Russel and Norvik: Artificial Intelligence, 2003, page 138'. Regions: West Australia (WA), South Australia (SA), etc.

$WA, NT, SA, Q, NSW, V, T \in \{red, green, yellow\}$

$WA \neq NT,\ WA \neq SA,\ NT \neq SA,\ SA \neq Q \ldots$

There exist several solutions.

Cryptarithmetic Puzzles

Each letter stands for a distinct digit. Find a substitution for the letters such that the sum is correct. No leading zeros allowed.

$$\begin{array}{cccccc} & & S & E & N & D \\ + & & M & O & R & E \\ \hline & M & O & N & E & Y \end{array}$$

Model: S,E,N,D,M,O,R,Y $\in \{0, 1, \ldots, 9\}$

all_different{S,E,N,D,M,O,R,Y}

1000S + 100E + 10N + D + 1000M + 100O + 10R + E =

10000M + 1000O + 100N + 10E + Y

Homework: TWO+TWO=FOUR.

The N-Queens Problem

Place N queens on a N×N chess board so that no queen threatens any of the other queens.

Model for 8 queens: $x_{11}, \ldots, x_{88} \in \{0, 1\}$

$$\sum_{j=1}^{8} x_{ij} = 1 \; \forall i$$

$$\sum_{i=1}^{8} x_{ij} = 1 \; \forall j$$

$$\sum_{(ij) \in D_k} x_{ij} \leq 1 \; \forall \text{ diagonals } D_k$$

Note: Binary variables and linear constraints!

How to solve a CSP?

CSP's are usually (NP-)hard problems.

- **Search**
  - Backtracking
- **Constraint propagation**
  - iterative process
  - process each constraint separately
  - domain reduction by using *consistency techniques*

For best results, combine search with the 'right' amount of propagation.

Consistency Techniques

- consider one constraint and its involved variables at a time
- directional
- goal: remove all values from the domain of a variable that do not have *support* in the other variable's domains
- *node consistency* (unary constraints)
- *arc consistency* (binary constraints) [AC-3 algorithm]
- *path consistency, k-consistency, strong k-consistency*
- *generalized arc consistency* for global constraints, e.g. all_different (bipartite matching problem)

Example: $X \in \{1, \ldots, 4\}$, $Y \in \{2, \ldots, 6\}$, $2X = Y$.

Domain reduction gives $X \in \{1, 2, 3\}$, $Y \in \{2, 4, 6\}$.

Important Issues for CSP Algorithms

- **Variable ordering** rules: *first-fail* (most constraint variable first) *largest degree first* (constraint graph)
- **Value ordering** rules: *least-constraining value first*
- Good variable and value orderings often based on experience.
- **Modelling aspects:** global constraints often give stronger propagation than binary constraints, but are more time-consuming. Choice of variables/constraints that result in efficient solution process.
- **Advantages:** modular structure, specialized fast algorithms for special constraints, etc.
- **Problems:** Cannot handle cost; modeler sometimes needs to implement own filtering algorithms.

  Trashing due to bad branching.

Optimization i CP

Constraint Optimization Problem (COP): a CSP with an objective function.

Solution approach to COP (minimization problem): Add a constraint of the form $objective \leq K$ and solve the CSP. If there is a solution, decrease K, otherwise increase K, and resolve ...

Problems: Unclear how to find the optimal K in an efficient way. Potentially very time consuming. Does not use the structure of the problem.

Cost Propagation

(Licentiate thesis by Birgit Grohe)

- Numerical propagation for optimization problems.
- Combines ideas from LP/ILP and CP.
- Generalizes domain reduction.
- Same structure as CP's constraint propagation, but propagates the costs themselves.
- Efficient use of cost information and the structure of the problem.
- ...