Lecture 15, Stochastic Programming— Optimization under Uncertainty

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Deterministic optimization

- Paramaters and conditions are assumed to be known

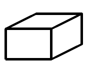
Optimization under uncertainty

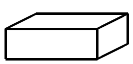
- Decisions must be made before all information is known for sure
- Examples: large investments, hydro and wind power production, maintenance planning (Ass 2), resource theory (Ass 1), ...
- Consider as much information as possible
- Consider also the decisions to be made later—when more information is known
- Uncertain data is represented by (discrete) probability distributions
- The size of the optimization problem *increases a lot* when uncertainties are considered explicitly

The LEGO furniture factory revisited

Small pieces

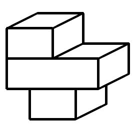
Purchase price: €100





Large pieces

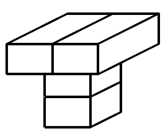
Purchase price: €200



Chair

Demand: $50 + \xi_2$

Sales price: $(5 + \eta_2)\cdot$ €100



Table

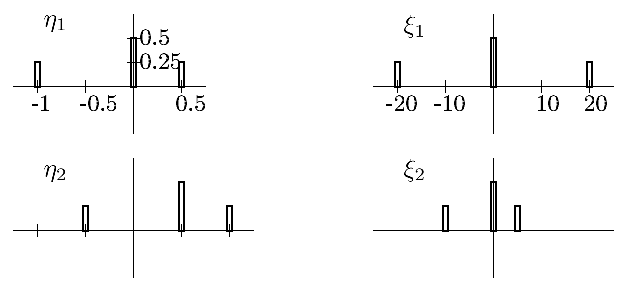
Demand: $100 + \xi_1$

Sales price: $(8 + \eta_1)\cdot$ €100

The *stochastic parameters* η_1, η_2, ξ_1, and ξ_2 are assumed to have discrete probability functions

Discrete probability functions

With probabilities $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{1}{4}$, respectively, the values $\eta_1 \in \{-1, 0, 0.5\}$, $\eta_2 \in \{-0.5, 0.5, 1\}$, $\xi_1 \in \{-20, 0, 20\}$, $\xi_2 \in \{-10, 0, 5\}$ are achieved



Assumption here: Low/medium/high demand levels correspond to low/medium/high price levels
⇒ The four stochastic parameters are *dependent* ⇒ Three scenarios

Definition of variables

- The demand and selling price is *not known* when the purchase of pieces is done. The purchase budget is 80000 €.
- Variabler:

$$
\begin{aligned}
x_1 &= \text{\# of large pieces purchased} \\
x_2 &= \text{\# of small pieces purchased} \\
y_1 &= \text{\# tables produced} \\
y_2 &= \text{\# chairs produced} \\
v_1 &= \text{\# tables sold} \\
v_2 &= \text{\# chairs sold}
\end{aligned}
$$

- The values of the purchase variables x_1 and x_2 must be decided before the demand and selling prices are known
- Production (y_1 and y_2) and sales (v_1 and v_2) are decided on later.

Mathematical model

Minimize *purchase costs* plus *production costs* minus *sales revenues*

$$
\begin{array}{llllllll}
\text{minimize}_{x,y,v} & \multicolumn{7}{l}{z = 100 \cdot [2x_1 + x_2 + y_1 - (8+\eta_1)v_1 + 0.5y_2 - (5+\eta_2)v_2]} \\
\text{subject to} & 2x_1 & +x_2 & & & \leq 800 & & \text{budget} \\
 & x_1 & & -2y_1 & -y_2 & \geq 0 & & \text{(large pieces used} \leq \text{purchased)} \\
 & & x_2 & -2y_1 & -2y_2 & \geq 0 & & \text{(small pieces used} \leq \text{purchased)} \\
 & & & y_1 - v_1 & & \geq 0 & & \text{(tables sold} \leq \text{produced)} \\
 & & & & y_2 - v_2 & \geq 0 & & \text{(chairs sold} \leq \text{produced)} \\
 & & & v_1 & & \leq 100 + \xi_1 & & \text{(sold} \leq \text{demand)} \\
 & & & & v_2 & \leq 50 + \xi_2 & & \text{(sold} \leq \text{demand)} \\
 & x_1, & x_2, & y_1, v_1 & y_2, v_2 & \geq 0 & \text{(integer)} &
\end{array}
$$

Deterministic (expected value) solution

Assume that the stochastic parameters attain their respective expected values:

$$\mathsf{E}(\eta_1) = \tfrac{1}{4} \cdot (-1) + \tfrac{1}{2} \cdot 0 + \tfrac{1}{4} \cdot 0.5 = -0.125$$

$$\mathsf{E}(\eta_2) = \tfrac{1}{4} \cdot (-0.5) + \tfrac{1}{2} \cdot 0.5 + \tfrac{1}{4} \cdot 1 = 0.375$$

$$\mathsf{E}(\xi_1) = \tfrac{1}{4} \cdot (-20) + \tfrac{1}{2} \cdot 0 + \tfrac{1}{4} \cdot 20 = 0$$

$$\mathsf{E}(\xi_2) = \tfrac{1}{4} \cdot (-10) + \tfrac{1}{2} \cdot 0 + \tfrac{1}{4} \cdot 5 = -1.25$$

Replace the stochastic parameters in the mathematical model by their expeced values.

The expected value solution

$$
\begin{array}{llllllll}
\text{minimize}_{x,y,z} & \multicolumn{7}{l}{z = 100 \cdot [2x_1 + x_2 + y_1 - 7.875v_1 + 0.5y_2 - 5.375v_2]} \\
\text{subject to} & 2x_1 & +x_2 & & & \leq 800 & \\
 & x_1 & & -2y_1 & -y_2 & \geq 0 & \\
 & & x_2 & -2y_1 & -2y_2 & \geq 0 & \\
 & & & y_1 - v_1 & & \geq 0 & \\
 & & & & y_2 - v_2 & \geq 0 & \\
 & & & v_1 & & \leq 100 & \\
 & & & & v_2 & \leq 48.75 & \\
 & x_1, & x_2, & y_1, v_1, & y_2, v_2 & \geq 0 & \text{(integer)}
\end{array}
$$

Solution: $x_1 = 248.75$, $x_2 = 297.5$, $y_1 = v_1 = 100$, $y_2 = v_2 = 48.75$, $z = -13016$ (minus the profit)

Deterministic solution: Purchase ≈ 249 large and ≈ 298 small pieces. Produce and sell ≈ 100 tables and ≈ 49 chairs.

Profit: €13016.

OBS: Infeasible at the lowest demand (80 tables and 40 chairs)!

A hedging deterministic optimization model

$$
\begin{array}{llllllll}
\text{minimize}_{x,y,z} & z = 100 \cdot [2x_1 & +x_2 & +y_1 - 7v_1 & +0.5y_2 - 4.5v_2] & & \\
\text{subject to} & 2x_1 & +x_2 & & & \leq 800 & \\
& x_1 & & -2y_1 & -y_2 & \geq 0 & \\
& & x_2 & -2y_1 & -2y_2 & \geq 0 & \\
& & & y_1 - v_1 & & \geq 0 & \\
& & & & y_2 - v_2 & \geq 0 & \\
& & & v_1 & & \leq 80 & \\
& & & & v_2 & \leq 40 & \\
& x_1, & x_2, & y_1, v_1, & y_2, v_2 & \geq 0 & \text{(integer)}
\end{array}
$$

Solution: $x_1 = 152.1$, $x_2 = 181.8$, $y_1 = v_1 = 61.22$, $y_2 = v_2 = 29.67$, $z = 0$

Deterministic solution: Purchase ≈ 152 large and ≈ 182 small pieces. Produce and sell ≈ 61 tables and ≈ 30 chairs.

Profit: €0

A stochastic optimization model

- **Step 1:** The purchase decision takes the possible outcomes of demand and selling prices into consideration, with their respective probabilities, and the decisions on production (and sales) that will be made later on.
- Three different *scenarios*: low/medium/high level of prices and demand: $\eta^1 = (-1, -0.5)$, $\xi^1 = (-20, -10)$, $\eta^2 = (0, 0.5)$, $\xi^2 = (0, 0)$, $\eta^3 = (0.5, 1)$, $\xi^3 = (20, 5)$
- **Step 2:** When the decisions on production (and sales) are made, the levels of prices and demand are assumed to be known
  Also the decided purchase of raw material is known

$\Rightarrow$ Perform as good as possible with respect to the outcome of the stochastic parameters and the decisions already made (recourse)

The first step decision

- Minimize the purchase cost minus *expected future profit*
- Decide on how many pieces to purchase (x)
- Consider the possible future outcomes of the demand (ξ) and price (η) levels and the production ($y(x, \xi, \eta)$) and sales ($v(x, \xi, \eta)$) decisions

$$
\begin{array}{llll}
\text{minimize}_x & z = 100 \cdot [2x_1 + x_2 & - \mathsf{E}_{\xi,\eta} Q(x, \xi, \eta)] & \text{(convex in } x\text{)} \\
\text{subject to} & 2x_1 + x_2 & \leq 800 & \text{purchase} \leq \text{budget} \\
& x_1, x_2 & \geq 0 & \text{(integer)}
\end{array}
$$

- $\mathsf{E}_{\xi,\eta} Q(x, \xi, \eta)]$ denotes the expected value of the future profit, which is computed in step two

The second step decision

- Maximize future profit (sales revenues minus production costs)
- Decide on production and sales for each outcome of the price (η) and demand (ξ) and for each purchase decision (x)

$$
Q(x, \xi, \eta) = \left(\begin{array}{lllll}
\text{maximize}_{y,v} & \multicolumn{4}{l}{-y_1 + (8 + \eta_1)v_1 - 0.5y_2 + (5 + \eta_2)v_2} \\
\text{subject to} & 2y_1 & +y_2 & \leq x_1 & \\
& 2y_1 & +2y_2 & \leq x_2 & \\
& v_1 & & \leq y_1 & \\
& & v_2 & \leq y_2 & \\
& v_1 & & \leq 100 + \xi_1 & \\
& & v_2 & \leq 50 + \xi_2 & \\
& y_1, v_1 & y_2, v_2 & \geq 0 & \text{(integer)}
\end{array} \right)
$$

Expected future profits

$$\mathsf{E}_{\xi,\eta}Q(x,\xi,\eta) = \frac{1}{4}Q(x,\xi^1,\eta^1) + \frac{1}{2}Q(x,\xi^2,\eta^2) + \frac{1}{4}Q(x,\xi^3,\eta^3) =$$

$$\frac{1}{4}\begin{pmatrix} \max -y_1^1 + 7v_1^1 - 0.5y_2^1 + 4.5v_2^1 & \\ \text{subject to} & 2y_1^1 + y_2^1 \le x_1 \\ & 2y_1^1 + 2y_2^1 \le x_2 \\ & v_1^1 \le \min\{y_1^1, 80\} \\ & v_2^1 \le \min\{y_2^1, 40\} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \max -y_1^2 + 8v_1^2 - 0.5y_2^2 + 5.5v_2^2 & \\ \text{subject to} & 2y_1^2 + y_2^2 \le x_1 \\ & 2y_1^2 + 2y_2^2 \le x_2 \\ & v_1^2 \le \min\{y_1^2, 100\} \\ & v_2^2 \le \min\{y_2^2, 50\} \end{pmatrix}$$

$$+\frac{1}{4}\begin{pmatrix} \max -y_1^3 + 8.5v_1^3 - 0.5y_2^3 + 6v_2^3 & \\ \text{subject to} & 2y_1^3 + y_2^3 \le x_1 \\ & 2y_1^3 + 2y_2^3 \le x_2 \\ & v_1^3 \le \min\{y_1^3, 120\} \\ & v_2^3 \le \min\{y_2^3, 55\} \end{pmatrix} \qquad \left(v_j^s \ge 0 \text{ and } y_j^s \ge 0, s = 1, 2, 3\right)$$

A deterministic equivalent model

$$\begin{array}{ll} \text{minimize} & z = 2x_1 + x_2 + \frac{1}{4}\left(y_1^1 - 7v_1^1 + 0.5y_2^1 - 4.5v_2^1\right) \\ & \quad + \frac{1}{2}\left(y_1^2 - 8v_1^2 + 0.5y_2^2 - 5.5v_2^2\right) + \frac{1}{4}\left(y_1^3 - 8.5v_1^3 + 0.5y_2^3 - 6v_2^3\right) \end{array}$$

$$\begin{array}{llccccccccl} \text{subject to} & 2x_1 & +x_2 & & & & & & & \le 800 \\ & -x_1 & & 2y_1^1 & +y_2^1 & & & & & \le 0 \\ & & -x_2 & 2y_1^1 & +2y_2^1 & & & & & \le 0 \\ & & & v_1^1 & & & & & & \le \min\{y_1^1, 80\} \\ & & & & v_2^1 & & & & & \le \min\{y_2^1, 40\} \\ & -x_1 & & & & 2y_1^2 & +y_2^2 & & & \le 0 \\ & & -x_2 & & & 2y_1^2 & +2y_2^2 & & & \le 0 \\ & & & & & v_1^2 & & & & \le \min\{y_1^2, 100\} \\ & & & & & & v_2^2 & & & \le \min\{y_2^2, 50\} \\ & -x_1 & & & & & & 2y_1^3 & +y_2^3 & \le 0 \\ & & -x_2 & & & & & 2y_1^3 & +2y_2^3 & \le 0 \\ & & & & & & & v_1^3 & & \le \min\{y_1^3, 120\} \\ & & & & & & & & v_2^3 & \le \min\{y_2^3, 55\} \\ & \multicolumn{8}{l}{x_1, \quad x_2,\ y_1^1, y_2^1,\ v_1^1, v_2^1,\ y_1^2, y_2^2,\ v_1^2, v_2^2,\ y_1^3, y_2^3\ v_1^3, v_2^3} & \ge 0 \text{ (heltal)} \end{array}$$

Solution to the optimization model that considers the uncertainty

Solution: $z = -10687$, $\Rightarrow$ Expected profit: €10687
$x = (200, 250)$, $y^1 = v^1 = (80, 40)$, $y^2 = v^2 = y^3 = v^3 = (75, 50)$

- If scenario 1 occurs (low) the profit becomes:
  $-100 \cdot (2 \cdot 200 + 250 + 80 + 0.5 \cdot 40 - 7 \cdot 80 - 4.5 \cdot 40) =$ €-1000
- If scenario 2 occurs (medium) the profit becomes:
  $-100 \cdot (2 \cdot 200 + 250 + 75 + 0.5 \cdot 50 - 8 \cdot 75 - 5.5 \cdot 50) =$ €12500
- If scenario 3 occurs (high) the profit becomes:
  $-100 \cdot (2 \cdot 200 + 250 + 75 + 0.5 \cdot 50 - 8.5 \cdot 75 - 6 \cdot 50) =$ €18750
- Expected profit: $\frac{-1000}{4} + \frac{12500}{2} + \frac{18750}{4} =$€10687

What if we should solve the deterministic model for the first decision and then adjust the solution to the actual scenario in the second step?

1. Solve the deterministic model $\Rightarrow \overline{x}_1 = 248.75$, $\overline{x}_2 = 297.5$
2. Compute the future profits $Q(x, \xi, \eta)$ for each of the three scenarios when $x = \overline{x}$
3. The expected value of the expected value (deterministic) solution is: $\quad z = 100 \cdot [2\overline{x}_1 + \overline{x}_2 - \mathsf{E}_{\xi,\eta}Q(\overline{x}, \xi, \eta)]$ (see next page)
4. The second step solutions become (three different scenarios):
   $y^1 = v^1 = (80, 40)$ $y^2 = v^2 = y^3 = v^3 = (100, 48.75)$
   Optimal objective value: $z =$ -9140
   $\Rightarrow$ Expected profit of expected value solution: €9140
5. Value of the stochastic solution: €10687 $-$ €9140 $=$ €1547

Expected future profits when $x = \overline{x} = (248.75, 297.5)$

$$\mathsf{E}_{\xi,\eta} Q(\overline{x},\xi,\eta) = \frac{1}{4}Q(\overline{x},\xi^1,\eta^1) + \frac{1}{2}Q(\overline{x},\xi^2,\eta^2) + \frac{1}{4}Q(\overline{x},\xi^3,\eta^3) =$$

$$\frac{1}{4}\begin{pmatrix} \max & -y_1^1 + 7v_1^1 - 0.5y_2^1 + 4.5v_2^1 \\ \text{subject to} & 2y_1^1 + y_2^1 \le 248.75 \\ & 2y_1^1 + 2y_2^1 \le 297.5 \\ & v_1^1 \le \min\{y_1^1, 80\} \\ & v_2^1 \le \min\{y_2^1, 40\} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \max & -y_1^2 + 8v_1^2 - 0.5y_2^2 + 5.5v_2^2 \\ \text{subject to} & 2y_1^2 + y_2^2 \le 248.75 \\ & 2y_1^2 + 2y_2^2 \le 297.5 \\ & v_1^2 \le \min\{y_1^2, 100\} \\ & v_2^2 \le \min\{y_2^2, 50\} \end{pmatrix}$$

$$+\frac{1}{4}\begin{pmatrix} \max & -y_1^3 + 8.5v_1^3 - 0.5y_2^3 + 6v_2^3 \\ \text{subject to} & 2y_1^3 + y_2^3 \le 248.75 \\ & 2y_1^3 + 2y_2^3 \le 297.5 \\ & v_1^3 \le \min\{y_1^3, 120\} \\ & v_2^3 \le \min\{y_2^3, 55\} \end{pmatrix} \qquad \left(v_j^s \ge 0 \text{ and } y_j^s \ge 0, s = 1,2,3\right)$$