MVE165, Applied Optimization
Lecture 2

2008–04–03

The simplex method for linear programs

- Every linear program can be reformulated such that:
  - all constraints are expressed as equalities with non-negative right hand sides and
  - all variables are non-negative
- These requirements streamline the simplex method calculations
- Commercial solvers can handle also inequality constraints and "free" variables—the reformulations are automatically taken care of

The simplex method—reformulations

- The lego example:

$$\left[\begin{array}{rrrr} 2x_1 & +x_2 & \leq & 6 \\ 2x_1 & +2x_2 & \leq & 8 \\ & x_1, x_2 & \geq & 0 \end{array}\right] \Leftrightarrow \left[\begin{array}{rrrrrr} 2x_1 & +x_2 & +s_1 & & = & 6 \\ 2x_1 & +2x_2 & & +s_2 & = & 8 \\ & & & x_1, x_2, s_1, s_2 & \geq & 0 \end{array}\right]$$

- s_1 and s_2 are called *slack variables*—"fill out" the (positive) distances between the left and right hand sides
- *Surplus variable* s_3:

$$\left[\begin{array}{ccccc} x_1 & + & x_2 & \geq & 800 \\ & & x_1, x_2 & \geq & 0 \end{array}\right] \Leftrightarrow \left[\begin{array}{ccccccc} x_1 & + & x_2 & - & s_3 & = & 800 \\ & & & & x_1, x_2, s_3 & \geq & 0 \end{array}\right]$$

The simplex method—reformulations, cont.

- Non-negative right hand side:

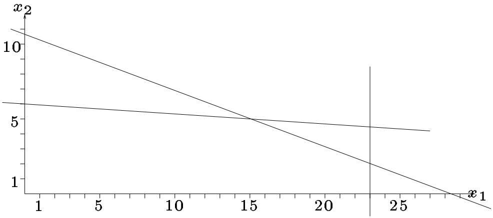
$$\begin{bmatrix} x_1 - x_2 & \leq -23 \\ x_1, x_2 & \geq 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} -x_1 + x_2 & \geq 23 \\ x_1, x_2 & \geq 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} -x_1 + x_2 - s_4 & = 23 \\ x_1, x_2, s_4 & \geq 0 \end{bmatrix}$$

- Free variables:

$$\begin{bmatrix} x_1 + x_2 & \leq 10 \\ x_1 & \geq 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} x_1 + x_2^1 - x_2^2 & \leq 10 \\ x_1, x_2^1, x_2^2 & \geq 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} x_1 + x_2^1 - x_2^2 + s_5 & = 10 \\ x_1, x_2^1, x_2^2, s_5 & \geq 0 \end{bmatrix}$$

Basic feasible solutions

- Consider m equations of n variables, where $m \leq n$
- Set $n - m$ variables to zero and solve (if possible) the remaining $(m \times m)$ system of equations
- If the solution is *unique*, it is callad a *basic* solution
- Such a solution corresponds to an intersection of m hyperplanes in $\mathbb{R}^m$ (feasible or infeasible)
- Each extreme point of the feasible set is an intersection of m hyperplanes with all variable values ≥ 0
- Basic feasible solution $\Leftrightarrow$ extreme point of the feasible set



Basic feasible solutions—the family dairy

- Constraints:

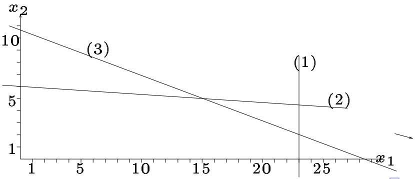
$$\begin{array}{rcrcll} x_1 & & & \le & 23 & (1) \\ 0.067x_1 & + & x_2 & \le & 6 & (2) \\ 3x_1 & + & 8x_2 & \le & 85 & (3) \\ & & x_1, x_2 & \ge & 0 & \end{array}$$

- Add slack variables:

$$\begin{array}{rrrrrll} x_1 & & +s_1 & & & = 23 & (1) \\ 0.067x_1 & +x_2 & & +s_2 & & = 6 & (2) \\ 3x_1 & +8x_2 & & & +s_3 & = 85 & (3) \\ & & & x_1, x_2, s_1, s_2, s_3 & & \ge 0 & \end{array}$$

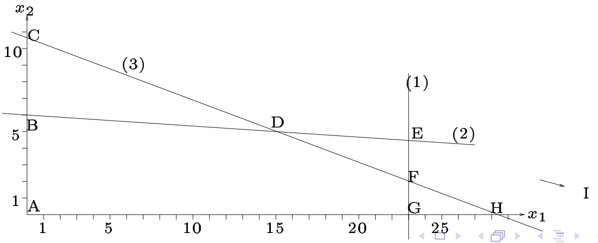
$m = 3$

$n = 5$

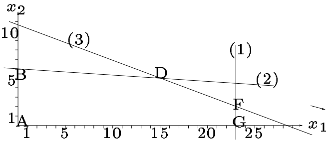


Basic and non-basic variables

basic variables	basic solution			non-basic variables $(0,0)$	point	feasible?
s_1, s_2, s_3	23	6	85	x_1, x_2	A	yes
s_1, s_2, x_1	$-5\frac{1}{3}$	$4\frac{1}{9}$	$28\frac{1}{3}$	s_3, x_2	H	no
s_1, s_2, x_2	23	$-4\frac{5}{8}$	$10\frac{5}{8}$	x_1, s_3	C	no
s_1, x_1, s_3	-67	90	-185	s_2, x_2	I	no
s_1, x_2, s_3	23	6	37	s_2, x_1	B	yes
x_1, s_2, s_3	23	$4\frac{7}{15}$	16	s_1, x_2	G	yes
x_2, s_2, s_3	-	-	-	s_1, x_1	-	-
x_1, x_2, s_1	15	5	8	s_2, s_3	D	yes
x_1, x_2, s_2	23	2	$2\frac{7}{15}$	s_1, s_3	F	yes
x_1, x_2, s_3	23	$4\frac{7}{15}$	$-19\frac{11}{15}$	s_1, s_2	E	no



Basic feasible solutions correspond to solutions to the system of equations that fulfil non-negativity



$$\left[\begin{array}{cccccc} x_1 & & +s_1 & & & = 23 \\ 0.067x_1 & +x_2 & & +s_2 & & = 6 \\ 3x_1 & +8x_2 & & & +s_3 & = 85 \end{array}\right]$$

where

A: $x_1 = x_2 = 0 \Rightarrow \left[\begin{array}{cccc} s_1 & & & = 23 \\ & s_2 & & = 6 \\ & & s_3 & = 85 \end{array}\right]$

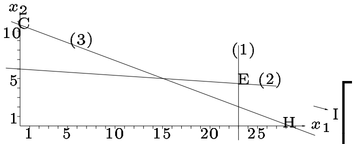
B: $x_1 = s_2 = 0 \Rightarrow \left[\begin{array}{cccc} & s_1 & & = 23 \\ x_2 & & & = 6 \\ 8x_2 & & +s_3 & = 85 \end{array}\right]$

D: $s_3 = s_2 = 0 \Rightarrow \left[\begin{array}{cccc} x_1 & & +s_1 & = 23 \\ 0.067x_1 & +x_2 & & = 6 \\ 3x_1 & +8x_2 & & = 85 \end{array}\right]$

F: $s_3 = s_1 = 0 \Rightarrow \left[\begin{array}{cccc} x_1 & & & = 23 \\ 0.067x_1 & +x_2 & +s_2 & = 6 \\ 3x_1 & +8x_2 & & = 85 \end{array}\right]$

G: $x_2 = s_1 = 0 \Rightarrow \left[\begin{array}{cccc} x_1 & & & = 23 \\ 0.067x_1 & +s_2 & & = 6 \\ 3x_1 & & +s_3 & = 85 \end{array}\right]$

Basic infeasible solutions correspond to solutions to the system of equations



$$\left[\begin{array}{ccccccc} x_1 & & +s_1 & & & = 23 \\ 0.067x_1 & +x_2 & & +s_2 & & = 6 \\ 3x_1 & +8x_2 & & & +s_3 & = 85 \end{array}\right]$$

where

H: $x_2 = s_3 = 0 \Rightarrow \left[\begin{array}{cccc} x_1 & +s_1 & & = 23 \\ 0.067x_1 & & +s_2 & = 6 \\ 3x_1 & & & = 85 \end{array}\right]$

C: $x_1 = s_3 = 0 \Rightarrow \left[\begin{array}{cccc} & s_1 & & = 23 \\ x_2 & & +s_2 & = 6 \\ 8x_2 & & & = 85 \end{array}\right]$

I: $s_2 = x_2 = 0 \Rightarrow \left[\begin{array}{cccc} x_1 & +s_1 & & = 23 \\ 0.067x_1 & & & = 6 \\ 3x_1 & & +s_3 & = 85 \end{array}\right]$

-: $s_1 = x_1 = 0 \Rightarrow \left[\begin{array}{cccc} & & 0 & = 23 \\ x_2 & +s_2 & & = 6 \\ 8x_2 & & +s_3 & = 85 \end{array}\right]$

E: $s_1 = s_2 = 0 \Rightarrow \left[\begin{array}{cccc} x_1 & & & = 23 \\ 0.067x_1 & +x_2 & & = 6 \\ 3x_1 & +8x_2 & +s_3 & = 85 \end{array}\right]$

Basic feasible solutions and the simplex method

- Express the m *basic* variables in terms of the $n-m$ *non-basic* variables
- Example: Start at $x_1 = x_2 = 0 \Rightarrow s_1,\ s_2,\ s_3$ are *basic*

$$\left[\begin{array}{cccccc} x_1 & & +s_1 & & & =23 \\ \frac{1}{15}x_1 & +x_2 & & +s_2 & & =6 \\ 3x_1 & +8x_2 & & & +s_3 & =85 \end{array}\right]$$

- Express s_1, s_2, and s_3 in terms of x_1 and x_2:

$$\left[\begin{array}{ccccc} s_1 & = & 23 & -x_1 & \\ s_2 & = & 6 & -\frac{1}{15}x_1 & -x_2 \\ s_3 & = & 85 & -3x_1 & -8x_2 \end{array}\right]$$

- Express the objective in terms of the *non-basic* variables:

$$z = 2x_1 + 3x_2 \qquad \Leftrightarrow \qquad z - 2x_1 - 3x_2 = 0$$

Basic feasible solutions and the simplex method

- The *first basic solution* can be represented as follows

$$
\begin{array}{cccccc|c}
-z & +2x_1 & +3x_2 & & & & = 0 & (0) \\
 & x_1 & & +s_1 & & & = 23 & (1) \\
 & \frac{1}{15}x_1 & +x_2 & & +s_2 & & = 6 & (2) \\
 & 3x_1 & +8x_2 & & & +s_3 & = 85 & (3)
\end{array}
$$

- The marginal values for increasing the non-basic variables x_1 and x_2 from zero are 2 and 3, respectively.

$\Rightarrow$ Choose x_2 — let x_2 *enter the basis* DRAW GRAPH!!

- One basic variable (s_1, s_2, or s_3) must leave the basis. Which one?
- The value of x_2 can increase until some basic variable reaches the value 0:

$$
\left.\begin{array}{ll}
(2): s_2 = 6 - x_2 \geq 0 & \Rightarrow x_2 \leq 6 \\
(3): s_3 = 85 - 8x_2 \geq 0 & \Rightarrow x_2 \leq 10\frac{5}{8}
\end{array}\right\} \Rightarrow \begin{array}{c} s_2 = 0 \text{ when} \\ x_2 = 6 \\ (\text{and } s_3 = 37) \end{array}
$$

- s_2 will leave the basis

Change basis through row operations

- Eliminate s_2 from the basis, let x_2 enter the basis using row operations:

$-z$	$+2x_1$	$+3x_2$				$=$	0	(0)
	x_1		$+s_1$			$=$	23	(1)
	$\frac{1}{15}x_1$	$+\mathbf{x_2}$		$+s_2$		$=$	6	(2)
	$3x_1$	$+8x_2$			$+s_3$	$=$	85	(3)
$-z$	$+\frac{9}{5}x_1$			$-3s_2$		$=$	-18	$(0){-}3{\cdot}(2)$
	x_1		$+s_1$			$=$	23	$(1){-}0{\cdot}(2)$
	$\frac{1}{15}x_1$	$+x_2$		$+s_2$		$=$	6	(2)
	$\frac{37}{15}x_1$			$-8s_2$	$+s_3$	$=$	37	$(3){-}8{\cdot}(2)$

- Corresponding basic solution: $s_1 = 23$, $x_2 = 6$, $s_3 = 37$.
- Nonbasic variables: $x_1 = s_2 = 0$
- The marginal value of x_1 is $\frac{9}{5} > 0$. Let x_1 enter the basis
- Which should leave? s_1, x_2, or s_3?

Change basis ...

$-z$	$+\frac{9}{5}x_1$			$-3s_2$		$=$	-18	(0)
	x_1		$+s_1$			$=$	23	(1)
	$\frac{1}{15}x_1$	$+x_2$		$+s_2$		$=$	6	(2)
	$\mathbf{\frac{37}{15}x_1}$			$-8s_2$	$+s_3$	$=$	37	(3)

- The value of x_1 can increase until some basic variable reaches the value 0:

$$\left.\begin{array}{lll} (1): s_1 = 23 - x_1 \geq 0 & \Rightarrow & x_1 \leq 23 \\ (2): x_2 = 6 - \frac{1}{15}x_1 \geq 0 & \Rightarrow & x_1 \leq 90 \\ (3): s_3 = 37 - \frac{37}{15}x_1 \geq 0 & \Rightarrow & x_1 \leq 15 \end{array}\right\} \Rightarrow \begin{array}{c} s_3 = 0 \text{ when} \\ x_1 = 15 \end{array}$$

- x_1 enters the basis and s_3 will leave the basis
- Perform row operations:

$-z$				$+2.84s_2$	$-0.73s_3$	$=$	-45	$(0)-(3)\cdot\frac{15}{37}\cdot\frac{9}{5}$
			s_1	$+3.24s_2$	$-0.41s_3$	$=$	8	$(1)-(3)\cdot\frac{15}{37}$
		x_2		$+1.22s_2$	$-0.03s_3$	$=$	5	$(2)-(3)\cdot\frac{15}{37}\cdot\frac{1}{1}$
	x_1			$-3.24s_2$	$+0.41s_3$	$=$	15	$(3)\cdot\frac{15}{37}$

Change basis ...

$-z$			$+2.84s_2$	$-0.73s_3$	$= -45$	(0)
		s_1	$+\mathbf{3.24s_2}$	$-0.41s_3$	$= 8$	(1)
	x_2		$+1.22s_2$	$-0.03s_3$	$= 5$	(2)
x_1			$-3.24s_2$	$+0.41s_3$	$= 15$	(3)

- Let s_2 enter the basis (marginal value > 0)
- The value of s_2 can increase until some basic variable $= 0$:

$$\left.\begin{array}{ll} (1): s_1 = 8 - 3.24s_2 \geq 0 & \Rightarrow s_2 \leq 2.47 \\ (2): x_2 = 5 - 1.22s_2 \geq 0 & \Rightarrow s_2 \leq 4.10 \\ (3): x_1 = 15 + 3.24s_2 \geq 0 & \Rightarrow s_2 \geq -4.63 \end{array}\right\} \Rightarrow \begin{array}{c} s_1 = 0 \text{ when} \\ s_2 = 2.47 \end{array}$$

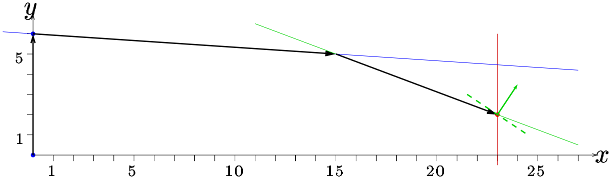
- s_2 enters the basis and s_1 will leave the basis
- Perform row operations:

$-z$		$-0.87s_1$		$-0.37s_3$	$= -52$	$(0)-(1)\cdot\frac{2.84}{3.24}$
		$0.31s_1$	$+s_2$	$-0.12s_3$	$= 2.47$	$(1)\cdot\frac{1}{3.24}$
	x_2	$-0.37s_1$		$+0.12s_3$	$= 2$	$(2)-(1)\cdot\frac{1.22}{3.24}$
x_1		$+s_1$			$= 23$	$(3)+(1)$

Optimal basic solution

$-z$			$-0.87s_1$		$-0.37s_3$	$=$ -52
			$0.31s_1$	$+s_2$	$-0.12s_3$	$=$ 2.47
		x_2	$-0.37s_1$		$+0.12s_3$	$=$ 2
	x_1		$+s_1$			$=$ 23

- No marginal value is positive. No improvement can be made
- The optimal basis is given by $s_2 = 2.47$, $x_2 = 2$, and $x_1 = 23$
- The variables s_1 and s_3 are non-basic
- The optimal value is $z = 52$



Summary of the solution course

basis	$-z$	x_1	x_2	s_1	s_2	s_3	RHS
$-z$	1	2	3	0	0	0	0
s_1	0	1	0	1	0	0	23
s_2	0	0.067	**1**	0	1	0	6
s_3	0	3	8	0	0	1	85
$-z$	1	1.80	0	0	-3	0	-18
s_1	0	1	0	1	0	0	23
x_2	0	0.07	1	0	1	0	6
s_3	0	**2.47**	0	0	-8	1	37
$-z$	1	0	0	0	2.84	-0.73	-45
s_1	0	0	0	1	**3.24**	-0.41	8
x_2	0	0	1	0	1.22	-0.03	5
x_1	0	1	0	0	-3.24	0.41	15
$-z$	1	0	0	-0.87	0	-0.37	-52
s_2	0	0	0	0.31	1	-0.12	2.47
x_2	0	0	1	-0.37	0	0.12	2
x_1	0	1	0	1	0	0	23

Summary of the simplex method

- **Optimality condition**: The *entering* variable in a maximization (minimization) problem should have the largest positive (negative) marginal value (reduced cost).

  The entering variable determines a direction in which the objective value increases (decreases).

  If all reduced costs are negative (positive), the current basis is optimal.

- **Feasibility condition**: The *leaving* variable is the one with smallest nonnegative ratio.

  Corresponds to the constraint that is "reached first"

Solve the lego problem using the simplex method!

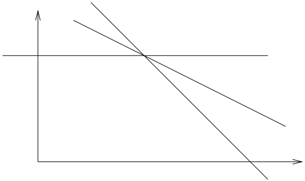
$$
\begin{array}{lrcrcrcr}
\text{maximize} & z & = & 1600x_1 & + & 1000x_2 & & \\
\text{subject to} & & & 2x_1 & + & x_2 & \leq & 6 \\
 & & & 2x_1 & + & 2x_2 & \leq & 8 \\
 & & & & & x_1,\ x_2 & \geq & 0
\end{array}
$$

Steps of the simplex method: see Taha, p. 100

Degeneracy

- If the smallest nonnegative ratio is zero, the value of a basic variable will become zero in the next iteration
- The solution *degenerate*
- The objective value will in fact *not* improve in this iteration
- There is a risk for *cycling* around (non-optimal) bases
- The reason is that there is a *redundant* constraint that "touches" the feasible set
- Example:

$$\begin{array}{rcrcl} x_1 & + & x_2 & \leq & 6 \\ & & x_2 & \leq & 3 \\ x_1 & + & 2x_2 & \leq & 9 \\ & & x_1,\ x_2 & \geq & 0 \end{array}$$



- There are computational rules to prevent from cycling

Unbounded solutions

- If all ratios are negative, the variable entering the basis may increase infinitely
- The feasible set is unbounded
- In a real applications this would probably be due to some incorrect assumption
- Example:

$$\begin{array}{rrrrl} \text{minimize} & z = & -x_1 & -2x_2 & \\ \text{subject to} & & -x_1 & +x_2 & \leq 2 \\ & & -2x_1 & +x_2 & \leq 1 \\ & & & x_1,\ x_2 & \geq 0 \end{array}$$

DRAW GRAPH!!

Unbounded solutions

- A feasible basis is given by $x_1 = 1$, $x_2 = 3$, with corresponding tableau:
  Homework: Find this basis using the simplex method.

basis	$-z$	x_1	x_2	s_1	s_2	RHS
$-z$	1	0	0	5	-3	7
x_1	0	1	0	1	-1	1
x_2	0	0	1	2	-1	3

- Entering variable is s_2
- Row 1: $x_1 = 1 + s_2 \geq 0 \Rightarrow s_2 \geq -1$
- Row 2: $x_2 = 3 + s_2 \geq 0 \Rightarrow s_2 \geq -3$
- No leaving variable can be found, since no constraint will prevent s_2 from increasing infinitely

Starting solution—finding an initial base

- Example:

$$\begin{array}{lrrrl} \text{minimize} & z = & 2x_1 & +3x_2 & \\ \text{subject to} & & 3x_1 & +2x_2 & = 14 \\ & & 2x_1 & -4x_2 & \geq 2 \\ & & 4x_1 & +3x_2 & \leq 19 \\ & & & x_1, x_2 & \geq 0 \end{array}$$

DRAW GRAPH!!

- Add slack and surplus variables

$$\begin{array}{lrrrrrl} \text{minimize} & z = & 2x_1 & +3x_2 & & & \\ \text{subject to} & & 3x_1 & +2x_2 & & & = 14 \\ & & 2x_1 & -4x_2 & -s_1 & & = 2 \\ & & 4x_1 & +3x_2 & & +s_2 & = 19 \\ & & & & & x_1, x_2, s_1, s_2 & \geq 0 \end{array}$$

- How to find an initial basis? Only s_2 is obvious!

Artificial variables

- Add artificial variables a_1 and a_2 to the first and second constraints, respectively
- Solve an artificial problem: minimize $a_1 + a_2$

$$
\begin{array}{llrrrrrrl}
\text{minimize} & w = & & & & & a_1 & +a_2 & \\
\text{subject to} & & 3x_1 & +2x_2 & & & +a_1 & & = 14 \\
 & & 2x_1 & -4x_2 & -s_1 & & & +a_2 & = 2 \\
 & & 4x_1 & +3x_2 & & +s_2 & & & = 19 \\
 & & & & & & & x_1, x_2, s_1, s_2, a_1, a_2 & \geq 0
\end{array}
$$

- This problem is called "phase one" problem
- An initial basis is given by $a_1 = 14$, $a_2 = 2$, and $s_2 = 19$:

basis	$-w$	x_1	x_2	s_1	s_2	a_1	a_2	RHS
$-w$	1	-5	2	1	0	0	0	-16
a_1	0	3	2	0	0	1	0	14
a_2	0	2	-4	-1	0	0	1	2
s_2	0	4	3	0	1	0	0	19

Find an initial solution using artificial variables

- x_1 enters $\Rightarrow a_2$ leaves (then $x_2 \Rightarrow s_2$, then $s_1 \Rightarrow a_1$)

basis	$-w$	x_1	x_2	s_1	s_2	a_1	a_2	RHS
$-w$	1	-5	2	1	0	0	0	-16
a_1	0	3	2	0	0	1	0	14
a_2	0	2	-4	-1	0	0	1	2
s_2	0	4	3	0	1	0	0	19
$-w$	1	0	-8	-1.5	0	0		-11
a_1	0	0	8	1.5	0	1		11
x_1	0	1	-2	-0.5	0	0		1
s_2	0	0	11	2	1	0		15
$-w$	1	0	0	-0.045	0.727	0		-0.091
a_1	0	0	0	0.045	-0.727	1		0.091
x_1	0	1	0	-0.136	0.182	0		3.727
x_2	0	0	1	0.182	0.091	0		1.364
$-w$	1	0	0	0	0			0
s_1	0	0	0	1	-16			2
x_1	0	1	0	0	-2			4
x_2	0	0	1	0	3			1

- A feasible basis is given by $x_1 = 4$, $x_2 = 1$, and $s_1 = 2$

Infeasible linear programs

- If the solution to the "phase one" problem has optimal value $= 0$, a feasible basis has been found

$\Rightarrow$ Start optimizing the original objective function z from this basis (homework)

- If the solution to the "phase one" problem has optimal value $w > 0$, no feasible points exists
- What would this mean for a real application?
- Alternative: M-method (Big-M method): Add the artificial variables to the original objective–with a large coefficient
  Example:

$$\text{minimize} \quad z = 2x_1 + 3x_2$$

$$\Rightarrow \qquad \text{minimize} \quad z_a = 2x_1 + 3x_2 + Ma_1 + Ma_2$$

Alternative optimal solutions

- Example:

$$\begin{array}{lrrrl} \text{minimize} & z = & 2x_1 & +4x_2 & \\ \text{subject to} & & x_1 & +2x_2 & \leq 5 \\ & & x_1 & +x_2 & \leq 4 \\ & & & x_1, x_2 & \geq 0 \end{array}$$

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- The extreme points $(0, \frac{5}{2})$ and $(3, 1)$ have the same optimal value $z = 10$
- All solutions that are positive linear (convex) combinations of these are optimal:

$$(x_1, x_2) = \alpha \cdot (0, \frac{5}{2}) + (1 - \alpha) \cdot (3, 1), \quad 0 \leq \alpha \leq 1$$

Recommended exercises

- Problem set 3.1A 1–3 & 5
- Problem set 3.1B 1 & 4 (only modelling part)
- Problem set 3.2A 1 & 4
- Problem set 3.3B 7 & 8
- Problem set 3.4A 9
- Problem set 3.4B 4 & 5
- Problem set 3.5A 2
- Problem set 3.5B 2
- Problem set 3.5C 2