Transportation Models

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An Example: MG Auto

(Taha, Example 5.1-1) MG Auto has three plants, LA, Detroit, New Orleans, and two distribution centers, Denver and Miami. Capacities of the plants are 1000, 1500 and 1200 cars. The demands are 2300 and 1400, respectively. The transportation cost between the plants and the centers is given below.

Find the cheapest shipping to satisfy the demand!

Taha, table 5.2 and figure 5.2

LP formulation of MG Auto

Taha, LP on p 192-193

Definition of the Transportation Model

Given m sources and n destinations, each represented as a **node**.

An **arc** (i, j) represents the connection between source node i and destination node j and is associated with a cost per unit c_{ij}. The (unknown) amount of shipped goods is denoted by x_{ij} for each arc. The amount of supply at source i is a_i and the amount of demand at destination j is b_j.

The objective is to find x_{ij} such that the total transportation cost is minimized while satisfying all supply and demand restrictions.

Taha, figure 5.1

Further Details about the Transportation Model

The Transportation model is a special case of Linear Programming.

For solving the model, we use representation in the **transportation tableau**.

Taha, table 5.3

If the total amount of demand does not equal the total amount of supply, it is necessary to **balance** the model by introducing dummy sources or dummy destinations, see Taha, Example 5.1-2, tables 5.4 and 5.5.

The constraint matrix has special properties which results in integer solutions also for LP-models.

More Examples: Production-Inventory-Control

(Taha, Example 5.2-1) Boralis produces backpacks for hikes. The demand occurs during March - June each year: 100, 200, 180 and 300 backpacks. The production capacity varies monthly: 50, 180,280 and 270 backpacks.

It is possible to store backpacks for 0.50 per backpack and month. It is also possible to deliver later, but this includes a penalty cost of 2 per backpack and month.

Boralis wants to determine the optimal production schedule.

Taha, table 5.12 and figure 5.3

The Transportation Algorithm

The algorithm follows the steps of the Simplex method. We use the transportation tableau instead of the simplex tableau. The special structure of the problem allows for simpler operations.

As with the Simplex method, we first need to find a starting solution. Then we iteratively improve this solution with pivot operations until the optimal solution is found.

Thus, the transportation algorithm can be divided into 3 steps:

1. Find a starting solution.
2. Find the *entering variable* by using the simplex optimality condition. If the optimality condition holds for all variables, stop. Else go to step 3.
3. Find the *leaving variable* by using the simples feasibility condition. Go to step 2.

Step 1: Finding a Starting Solution

- Set all $x_{ii} = 1$ and $x_{ij} = 0$ for $i \neq j$.
- Northwest-Corner method
- Least Cost method
- Vogel Approximation method (VAM)

Taha, Examples 5.3-1, 5.3-2, and 5.3-3

Step 2: Finding the Entering Variable

Taha: *Method of multipliers*. Or: *Reduced cost* $\bar{c}$ *computations*.

Goal: Compute the reduced costs of all non-basic variables and check the optimality condition ($\bar{c}_j \leq 0$ for all j).

To compute the $\bar{c}_N$, use that $\bar{c}_B = 0$. From $\bar{c}_B = 0$ we can conclude the value of the dual variables/multipliers.

Taha, Example 5.3-5, Figure 5.22

Step 3: Finding the Leaving Variable

One of the basic variables has to leave the basis. Select the variable according the the simplex feasibility condition ($x_{ij} \geq 0$).

Taha, Example 5.3-5, Table 5.23 - 5.25

LP-view of the Transportation Algorithm

$$\begin{array}{lrcl} \min & \sum_{ij} c_{ij} x_{ij} & & \\ \text{s.t.} & \sum_j x_{ij} & = & a_i \\ & \sum_i x_{ij} & = & b_j \\ & x_{ij} & \geq & 0 \end{array}$$

$$\begin{array}{lc} \max & \sum_i a_i u_i + \sum_j b_j v_j \\ \text{s.t.} & u_i + v_j \leq c_{ij} \\ & u_i, v_j \text{ unrestricted} \end{array}$$

The Assignment Model

The Assignment model is a special case of the Transportation model.

Given n persons and n jobs. Given further the cost c_{ij} of assigning person i to job j. Introduce binary variables, $x_{ij} = 1$ if person i does job j and $x_{ij} = 0$ otherwise. Find the cheapest assignment of persons to jobs such that all jobs are done.

$$
\begin{array}{lrcl}
\min & \sum_{ij} c_{ij} x_{ij} & & \\
\text{s.t.} & \sum_j x_{ij} & = & 1 \\
 & \sum_i x_{ij} & = & 1 \\
 & x_{ij} & \geq & 0
\end{array}
$$

An Example

(Taha, Example 5.4-1). Given 3 children, John, Karen and Terri, and given three tasks, mow, paint and wash. Given further a cost for each combination of child/task. How should the parents distribute the tasks to minimize the cost?

Taha, Tables 5.31 - 5.34

What if there are more tasks and children? Taha, tables 5.35 - 5.36.

A Solution Method for the Assignment Model

The Simplex method, the Transportation algorithm, or the *Hungarian method.*

1. From the matrix c_{ij}, subtract the row and the column minimum from each entry.
2. If the 0's allow an assignment solution, stop. Otherwise go to step 3.
3. (a) Draw the minimum number of horizontal and vertical lines in the modified matrix $\bar(c)_(ij)$.
   (b) Identify the uncovered entry with the smallest value, k.
   (c) Add K to all uncovered rows (columns) and subtract K from all covered columns (rows).