Chalmers University of Technology — MVE165
University of Gothenburg — MMG630
Mathematical Sciences — Applied Optimization
Optimization — Assignment information
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Assignment 2: Maintenance planning

Given below is a mathematical model for finding a maintenance schedule such that the costs of maintaining a system during a limited time period is at minimum. The system consists of several components with economic dependencies. The model is described in the notes of Lecture 10.

Implementations of the model in AMPL can be found on the course homepage:
`http://www.math.chalmers.se/Math/Grundutb/CTH/mve165/0809/`
Study the AMPL files carefully to get some hints before you start solving. Call AMPL/CPLEX using the command '`ampl uh.run`'. The file `uh.run` should be edited in order to solve the different instances of the model, as described in the exercises below.

To pass the assignment you should (in groups of two persons) `(i)` write a report (maximum six pages) that describes and discusses the issues presented in the exercises and questions below. We also ask you to estimate the number of hours you spent on this assignment and note this in your report.

The report should be e-mailed to `anstr@chalmers.se`
at the latest on Thursday 30 of April 2009 — (extended deadline).

You should then `(ii)` write an opposition (maximum 1/2 page) to another group's report which should be handed in
at the latest on Thursday 7 of May 2009 — (extended deadline).

The questions 1–3 below are mandatory. In addition, students aiming at grade 3 or G must answer *at least one* of the questions 4–6, while students aiming at grade 4, 5, or VG must answer *all* the questions.

The mathematical model

Sets and parameters

- $\mathcal{N}$ = the set of components in the system. `(in AMPL: Components)`
- T = the number of time steps in the planning period. `(in AMPL: T)`
- T_i = the life of a new component of type $i \in \mathcal{N}$ (measured in number of time steps). It is assumed that $2 \leq T_i \leq T - 1$. `(in AMPL: U)`
- c_{it} = the cost of a replacement component of type $i \in \mathcal{N}$ at time t (measured in €). For some instances it is assumed that c_{it} is constant over time, i.e., $c_{it} = c_i$, $t = 1, \ldots, T$. `(in AMPL: c)`

- d_t = the cost for a maintenance occasion at time t (measured in €). For some instances it is assumed that d_t is constant over time, i.e., $d_t = d$, $t = 1, \ldots, T$. `(in AMPL: d)`

Decision variables

- $x_{it} = \begin{cases} 1 & \text{if component } i \text{ is replaced at time } t, \\ 0 & \text{otherwise,} \end{cases} \qquad i \in \mathcal{N}, \quad t \in \{1, \ldots, T\}.$

- $z_t = \begin{cases} 1 & \text{if maintenance is made at time } t, \\ 0 & \text{otherwise,} \end{cases} \qquad t \in \{1, \ldots, T\}.$

The model

$$
\begin{aligned}
\text{minimize} \quad & \sum_{t=1}^{T} \left(\sum_{i \in \mathcal{N}} c_{it} x_{it} + d_t z_t \right), && && (1) \\
\text{subject to} \quad & \sum_{t=\ell+1}^{T_i+\ell} x_{it} \geq 1, && \ell = 0, \ldots, T - T_i, \quad i \in \mathcal{N}, && (2) \\
& x_{it} \leq z_t, && t = 1, \ldots, T, \quad i \in \mathcal{N}, && (3) \\
& x_{it}, z_t \in \{0, 1\}, && t = 1, \ldots, T, \quad i \in \mathcal{N}. && (4)
\end{aligned}
$$

Description of the model

(1) The objective is to minimize the total cost for the maintenance during the planning period (the time steps $1, \ldots, T-1$) `(in AMPL: Cost)`.

(2) Each component i must be replaced at least once within each T_i time steps `(in AMPL: ReplaceWithinLife)`.

(3) Components can only be replaced at maintenance occasions `(in AMPL: ReplaceOnlyAtMaintenance)`.

(4) All the variables are required to be binary.

Exercises to perform and questions to answer

1. (a) Solve the model (1)–(4) as implemented in `uh-small.mod` with data from `uh-small.dat` and with integer requirements on the variables x_{it} and z_t.

   (b) Relax the integrality on the variables x_{it} and resolve the model. Relax the integrality on all variables and resolve the model. Compare the solutions obtained and discuss their interpretations.

   (c) Add the constraint in `cgcut.mod` to the model `uh-small.mod` and solve (with data from `uh-small.dat`) with *no integrality requirements on the variables*. Compare the solution to those obtained in 1a and 1b above. Explain what happened.

2. Solve four instances of the model (1)–(4) as implemented in `uh-larger.mod` and `uh-larger.dat`. Let the cost of a maintenance occasion for all t be given by $d_t = 0.00001$, $d_t = 20$, $d_t = 100$, and $d_t = 10000$, respectively.

(a) Compare the number of maintenance occasions, the number of replaced components, and the total cost for each of these values of d_t.

(b) Draw illustrating maintenance schedules for each of the solutions.

(c) Discuss what is governing the compromises. What happens between $d_t = 100$ and $d_t = 10000$? Why? Find the smallest value of d_t for which the number of maintenance occasions is as small as possible. How is this value related to the replacement costs?

3. Solve the model (1)–(4) as implemented in `uh-larger.mod` and `uh-larger.dat` (with $d = 20$).

   (a) Vary the time horizon between $T = 50$ and $T = 300$ in steps of 25 (refine the step size if/where needed) and draw a graph of the solution time (in CPU seconds) as a function of T. (Use possibly a log-scale.)

   (b) Make an analogous graph for the case when the integrality requirements on the variables are relaxed, but let the upper limit on the time horizon be higher than in 3a. Denote whether the solutions found are integral.

   (c) Comment on the (complexity) properties of the functions in 3a and 3b.

4. This is a theoretical question concerning the model (1)–(4). Your answers and motivations should be valid for *any* data, but using the models previously implemented together with the data given may help your reasoning.

   For the questions 4a–4c below, consider the model (1)–(4) with time independent costs (i.e., $c_{it} = c_i$ and $d_t = d$ for $t \in \mathcal{T}$ and $i \in \mathcal{N}$, as in `uh-larger.mod`).

   (a) Does there always exist an optimal solution with $x_{iT} = 0$, $i \in \mathcal{N}$, and $z_T = 0$? If so, explain why. If not, provide a counter example.

   (b) Assume that $T_i = 1$ for some component $i \in \mathcal{N}$. What can be said about the solution to the model? How difficult is the problem to solve?

   (c) Assume that $T_k \geq T$ for some component $k \in \mathcal{N}$. What can be said about the solution to the model?

   (d) If we allow the costs to vary over time (as in `uh-smaller.mod`), what are then the answers to the questions 4a–4c?

5. Assume that it is required that the system (including all of its components) has a remaining life which is at least $r > 0$ time steps at the end of the planning period (i.e., at time $t = T$).

   (a) Add and/or modify constraints to/in the model to accomplish this and solve the resulting model. Start from the model in `uh-larger.mod` with data from `uh-larger.dat` (with $d = 20$ and $T = 155$). Verify that the solution fulfills the requirement stated.

   (b) For relevant values of r (at least two, at most five different values), compare the total cost for maintenance according to this schedule with that of the "original" one. Comment on the number of maintenance occasions and the number of replaced components and compare to the corresponding numbers from the "original" model.

   (c) Which values of r are relevant for this study and why?

6. An important application of maintenance planning concerns aircraft engines, which consist of several modules, each consisting of several components. This model is more complex than that defined by (1)–(4). Each time a specific

component is replaced, a cost has to be paid for "opening" the engine and an additional cost has to be paid for "removing and opening" the module in which the component to be replaced is located.

The data for the problem that we will consider is given in Tables 1 and 2 in addition to a cost of 10 for opening the engine and the planning horizon being $T = 50$. For instance, if the stator, the exhaust frame and the disc are replaced at the same time, the cost generated will be $10 + 5 + 12 + 20 + 15 + 25$.

component	module	life	replacement cost
stator	turbine	15	12
exhaust frame	turbine	8	20
roller	turbine	20	24
seal segment	after burner	12	14
case	after burner	10	19
nozzle segment	after burner	8	15
shaft	fan	11	12
disc	fan	25	25
blade	fan	30	10

Table 1: A list of components together with the modules to which they belong, their lives, and their replacement costs.

module	cost for removal
turbine	5
after burner	10
fan	15

Table 2: A list of all modules together with the cost of their removal.

(a) Create a model that minimizes the maintenance cost over the planning horizon of an aircraft engine consisting of several modules. Present this model using mathematical notation. Solve the model with the specific data provided above and present your results in a figure showing at which time steps maintenance is performed, which components are then replaced, and which modules are removed.

(b) Consider the model implemented in `uh-larger.mod`, let $d = 30$ ($\approx$ cost for opening the engine + average cost of removing modules), and use the data from Table 1 for the components' lives and the replacement costs, and use the time horizon $T = 50$. Solve this problem and present the solution graphically. Compare to the solution in 6a. What would be the cost of using this solution if the costs of opening up the modules would be calculated individually for each module, as modelled in 6a?

(c) Cplex uses the branch–and–bound algorithm. On what does it seem to spend most of the solution time: finding an optimal solution or verifying its optimality? Use your model and data from 6a and run it for the time horizon $T = 70$. Discuss your findings in terms of the duality gap.