MVE165/MMG630, Applied Optimization
Lecture 1

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Course contents and organization

Contents

- Applications of optimization
- Mathematical modelling
- Solution techniques – algorithms
- Software solvers

Organization

- Lectures – mathematical optimization theory
- Exercises – use solvers, oral examination (or report)
- Guest lectures – applications of optimization
- Assignments – modelling, use solvers, written reports, opposition & oral presentations
- Assignment work should be done in groups of two persons

Staff and homepage

Staff

- **Examiner/lecturer:** Ann-Brith Strömberg (anstr@chalmers.se, room L2087)
- **Substitute lecturers:** Peter Lindroth & Adam Wojciechowski
- **Guest lecturers:** Fredrik Hedenus (Energy and Environment), Michael Patriksson (Mathematical Sciences), Elin Svensson (Energy and Environment), and Caroline Olsson (Radiation Physics, Clinical Sciences)

Course homepage

`http://www.math.chalmers.se/Math/Grundutb/CTH/mve165/0809/`

- Contains details, information on assignmnents and exercises, deadlines, lecture notes, etc
- Will be updated every week during the course

Literature and examination

Literature

- R.L. Rardin: "Optimization in Operations Research", published by Prentice-Hall, New Jersey (1998)
- Found at Cremona (by the end of this week)

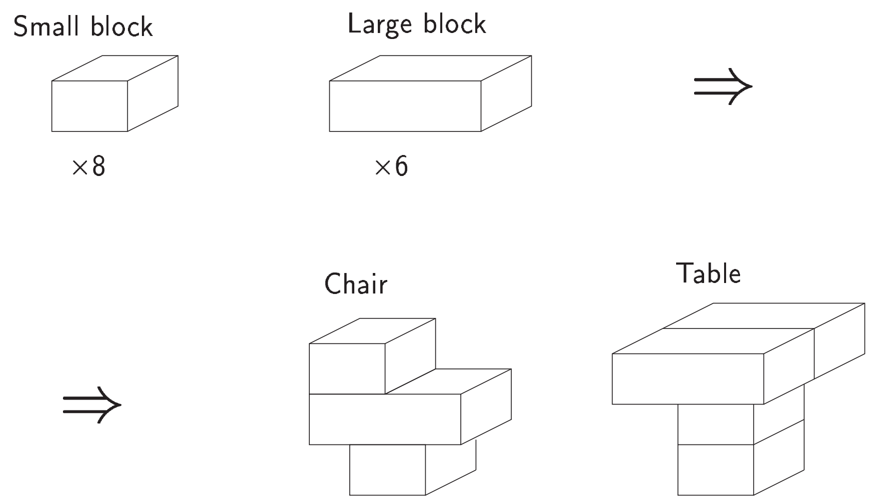
Examination

- Two correctly solved computer exercises (oral examination or written reports)
- Written reports of three assignments (1, 2, and 3a or 3b)
- A written opposition to Assignment 2
- An oral presentation of Assignment 3a or 3b
- To be able to receive a grade higher than 3 or G, the written reports and opposition as well as the oral presentation must be of high quality. Students aiming at grade 4, 5, or VG must also pass an oral exam

Overview of the lectures

- Linear programming, modelling, theory, solution methods, sensitivity analysis
- Optimization models that can be described as flows in networks, solution methods
- Discrete optimization models and solution methods
- Non-linear programming models, with(out) constraints, solution methods
- Multiple objective optimization
- Optimization under uncertainty
- Mixtures of the above

A manufacturing example: Produce tables and chairs from two types of blocks



Optimization

"Do something as good as possible"

- **Something:** Which are the decision alternatives?
- **Possible:** What restrictions are there?
- **Good:** What is a relevant optimization criterion?

A manufacturing example, continued

- A chair is assembled from one large and two small blocks
- A table is assembled from two blocks of each size
- Only 6 large and 8 small blocks are avaliable
- A table is sold at a revenue of 1600:-
- A chair is sold at a revenue of 1000:-
- Assume that all items produced can be sold and determine an optimal production plan

A mathematical optimization model

Something: Which are the decision alternatives? $\Rightarrow$ *Variables*

$$\begin{aligned} x_1 &= \text{number of tables produced and sold} \\ x_2 &= \text{number of chairs produced and sold} \end{aligned}$$

Possible: What restrictions are there? $\Rightarrow$ *Constraints*

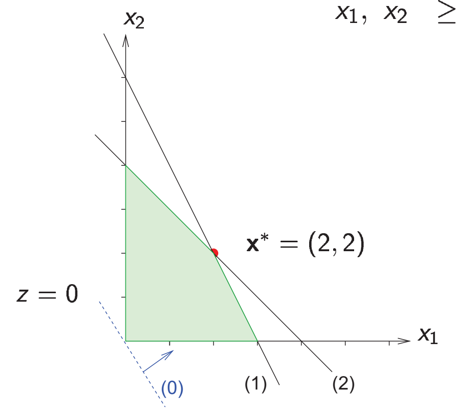
$$\begin{array}{rcrcll} 2x_1 & + & x_2 & \leq & 6 & \text{(6 large blocks)} \\ 2x_1 & + & 2x_2 & \leq & 8 & \text{(8 small blocks)} \\ & & x_1,\ x_2 & \geq & 0 & \text{(physical restrictions)} \\ & & (x_1,\ x_2 & & \text{integral}) & \text{(physical restrictions)} \end{array}$$

Good: What is a relevant optimization criterion? $\Rightarrow$ *Objective function*

$$\text{maximize} \quad z = 1600x_1 + 1000x_2 \qquad (z = \text{total revenue})$$

Geometrical solution of the model

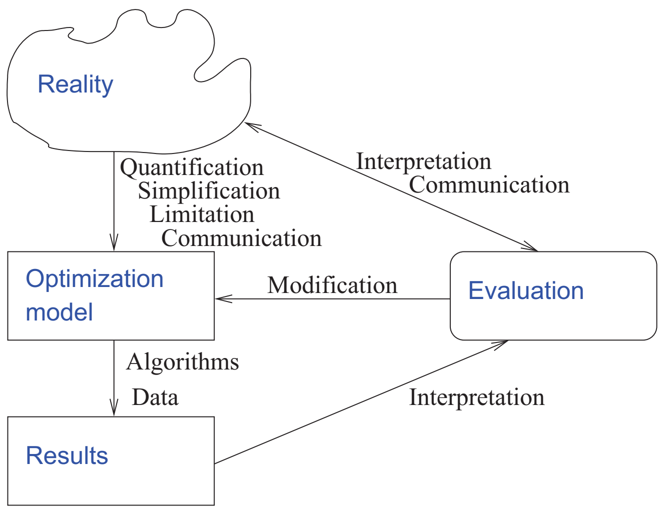
$$\begin{array}{llcrcrcll} \text{maximize} & z & = & 1600x_1 & + & 1000x_2 & & & (0) \\ \text{subject to} & & & 2x_1 & + & x_2 & \leq & 6 & (1) \\ & & & 2x_1 & + & 2x_2 & \leq & 8 & (2) \\ & & & & & x_1,\ x_2 & \geq & 0 & \end{array}$$



Solve the model using LEGO!

- Start at no production: $x_1 = x_2 = 0$
  Use the "best marginal profit" to choose the item to produce
  - x_1 has the highest marginal profit (1600:-/table)
    $\Rightarrow$ produce as many tables as possible
  - At $x_1 = 3$: no more large blocks left
- The marginal value of x_2 is now 200:- since taking apart one table ($-$1600:-) yields two chairs (+2000:-) $\Rightarrow$ 400:-/2 chairs
  - Increase x_2 maximally $\Rightarrow$ decrease x_1
  - At $x_1 = x_2 = 2$: no more small blocks
- The marginal value of x_1 is negative (to build one more table one has to take apart two chairs $\Rightarrow$ $-$400:-)
  The marginal value of x_2 is $-$600:- (to build one more chair one table must be taken apart)
  $\Longrightarrow$ Optimal solution: $x_1 = x_2 = 2$

Operations research—more than just mathematics



Concepts

- **Feasible solution:** satisfies all constraints
- **Optimal solution:** feasible AND objective function value as good as for every feasible solution
- **Sensitivity analysis:** how the solution depend on input parameters
- **Tractability:** Can the the model be solved in reasonable time?
- **Validity:** Does the conclusions drawn from the solution hold for the REAL problem
- **Operations research:** Always a tradeoff between validity of the model and its tractability to analysis

Optimization modelling: A production–inventory example

- Deliver windows over a six-month period
- Demand for each month: 100, 250, 190, 140, 220 & 110 units
- Production cost/window: 50 €,45 €, 55 €, 48 €, 52 € & 50 €
- Store a produced window from one month to the next at 8 €
- Meet the demands and minimize costs
- Find an optimal production schedule

More concepts

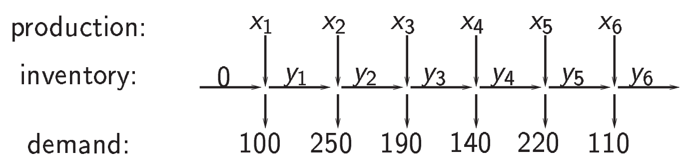
- **Optimal solution:** proven to be as good as any other feasible solution
- **Heuristic or approximate solution:** feasible, not guaranteed to be exactly optimal, quality measures can be computed
- **Deterministic optimization model:** All parameter values assumed known with certainty
- **Stochastic optimization model:** involves quantities known only in probability (optimization under uncertainty)
- **Multiple objective optimization:** typically no feasible solution exist that is optimal in ALL objectives, search for Pareto optimal solutions

Define the decision variables

$$x_i = \text{number of units produced in month } i = 1, \ldots, 6$$

$$y_i = \text{units left in the inventory at the end of month } i = 1, \ldots, 6$$

- The "flow" of windows can be illustrated as:



Define the limitations/constraints

- Each month:

initial inventory + production − ending inventory = demand

$$
\begin{array}{ccccccccl}
0 & + & x_1 & - & y_1 & = & 100 & & \\
y_1 & + & x_2 & - & y_2 & = & 250 & & \\
y_2 & + & x_3 & - & y_3 & = & 190 & & \\
y_3 & + & x_4 & - & y_4 & = & 140 & & \\
y_4 & + & x_5 & - & y_5 & = & 220 & & \\
y_5 & + & x_6 & - & y_6 & = & 110 & & \\
 & & x_i & , & y_i & \geq & 0, & & i = 1, \dots, 6
\end{array}
$$

Objective function: minimize the costs

- Production cost (€):

$$50\ x_1 + 45\ x_2 + 55\ x_3 + 48\ x_4 + 52\ x_5 + 50\ x_6$$

- Inventory cost (€):

$$8\ (y_1 + y_2 + y_3 + y_4 + y_5 + y_6)$$

- Objective:

$$
\begin{array}{ll}
\text{minimize} & 50x_1 + 45x_2 + 55x_3 + 48x_4 + 52x_5 + 50x_6 \\
 & +8(y_1 + y_2 + y_3 + y_4 + y_5 + y_6)
\end{array}
$$

A complete (general) optimization model

$$
\begin{array}{lrcll}
\text{minimize} & \multicolumn{3}{l}{\sum_{i=1}^{6} c_i x_i + 8 \sum_{i=1}^{6} y_i,} & \\
\text{subject to} & y_{i-1} + x_i - y_i & = & d_i, & i = 1, \dots, 6, \\
 & y_0 & = & 0, & \\
 & x_i, y_i & \geq & 0, & i = 1, \dots, 6,
\end{array}
$$

The vector of demand:

$$d = (d_i)_{i=1}^{6} = (100, 250, 190, 140, 220, 110)$$

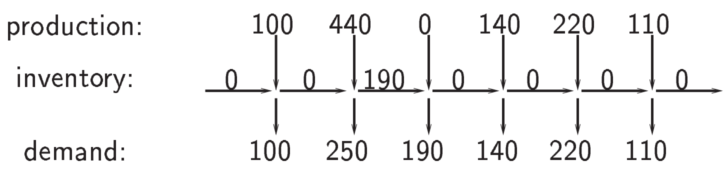
The vector of production costs:

$$c = (c_i)_{i=1}^{6} = (50, 45, 55, 48, 52, 50) \qquad .$$

An optimal solution—optimal production schedule

Optimal production: $x = (x_i)_{i=1}^{6} = (100, 440, 0, 140, 220, 110)$

Optimal inventory: $y = (y_i)_{i=0}^{6} = (0, 0, 190, 0, 0, 0, 0)$



The minimal total cost is 49980 €

Mathematical optimization models

$$\left[\begin{array}{llll} \text{minimize or maximize} & f(x_1,\ldots,x_n) & & \\ \text{subject to} & g_i(x_1,\ldots,x_n) & \left\{\begin{array}{c}\le\\=\\\ge\end{array}\right\} & b_i, \quad i=1,\ldots,m \end{array}\right]$$

- $x_1,\ldots,x_n$ are the decision variables
- f and $g_1,\ldots,g_m$ are given functions of the decision variables
- $b_1,\ldots,b_m$ are specified constant parameters
- The functions can be nonlinear, e.g. quadratic, exponential, logarithmic, non-analytic, ...
- In general, linear forms are more tractable than non-linear

Discrete/integer/binary modelling

- A variable is discrete if it can take only a countable set of values, e.g.,
  - Discrete variable: $x \in \{0, 4.4, 5.2, 8.0\}$
  - Integer variable: $x \in \{0, 1, 4, 5, 8\}$
- A binary variable can only take values 0 or 1 - all or nothing

  E.g., a wind-mill can produce electricity only if it is built
  - Let $y = 1$ if the mill is built, else $y = 0$
  - Capacity of a mill: C
  - Production $x \le C \cdot y$ (also limited by wind force etc.)
- In general, models with only continuous variables are more tractable than models with integrality/discrete requirements on variables, but exceptions exist!

Linear programming models

- The production inventory model is a linear program (LP) – all relations are described by linear forms
- A general linear program:

$$\left[\begin{array}{lcccl} \text{min or max} & c_1x_1+\ldots+c_nx_n & & & \\ \text{subject to} & a_{i1}x_1+\ldots+a_{in}x_n & \left\{\begin{array}{c}\le\\=\\\ge\end{array}\right\} & b_i, & i=1,\ldots,m \\ & x_j & \ge & 0, & j=1,\ldots,n \end{array}\right]$$

- The non-negativity constraints on x_j, $j = 1,\ldots,n$ are not necessary, but usually assumed (reformulation always possible)