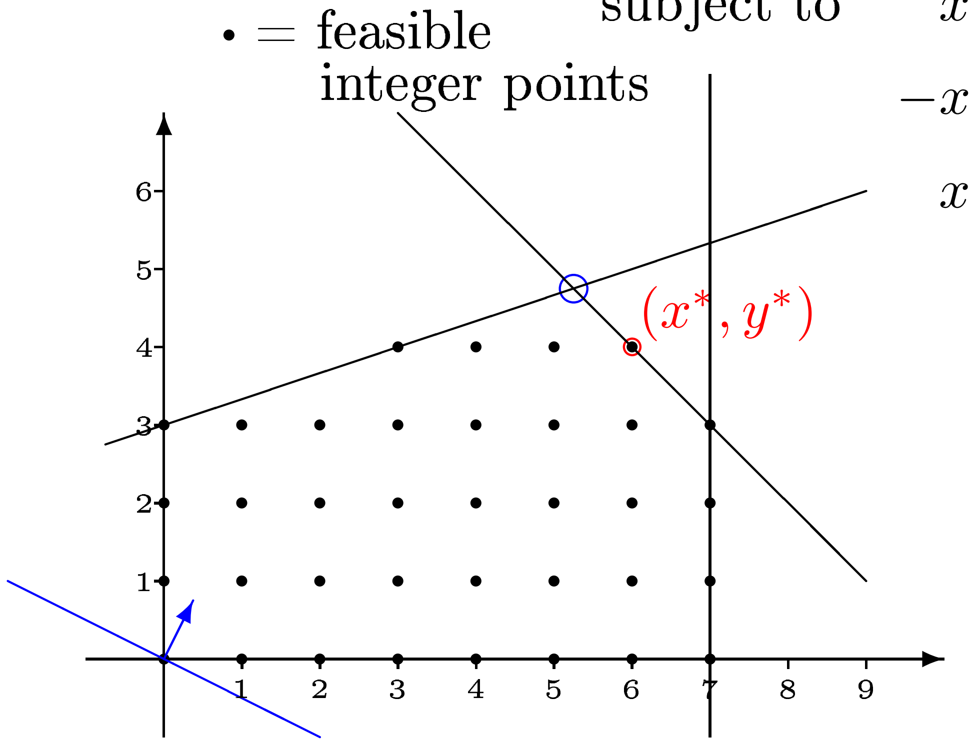
An integer linear program

$$\begin{array}{lrcrcl l}
\text{maximize} & x & + & 2y & & & \\
\text{subject to} & x & + & y & \leq & 10 & (1) \\
 & -x & + & 3y & \leq & 9 & (2) \\
 & x & & & \leq & 7 & (3) \\
 & & & x, y & \geq & 0 & (4,5) \\
 & & & x, y & & \text{integer} &
\end{array}$$



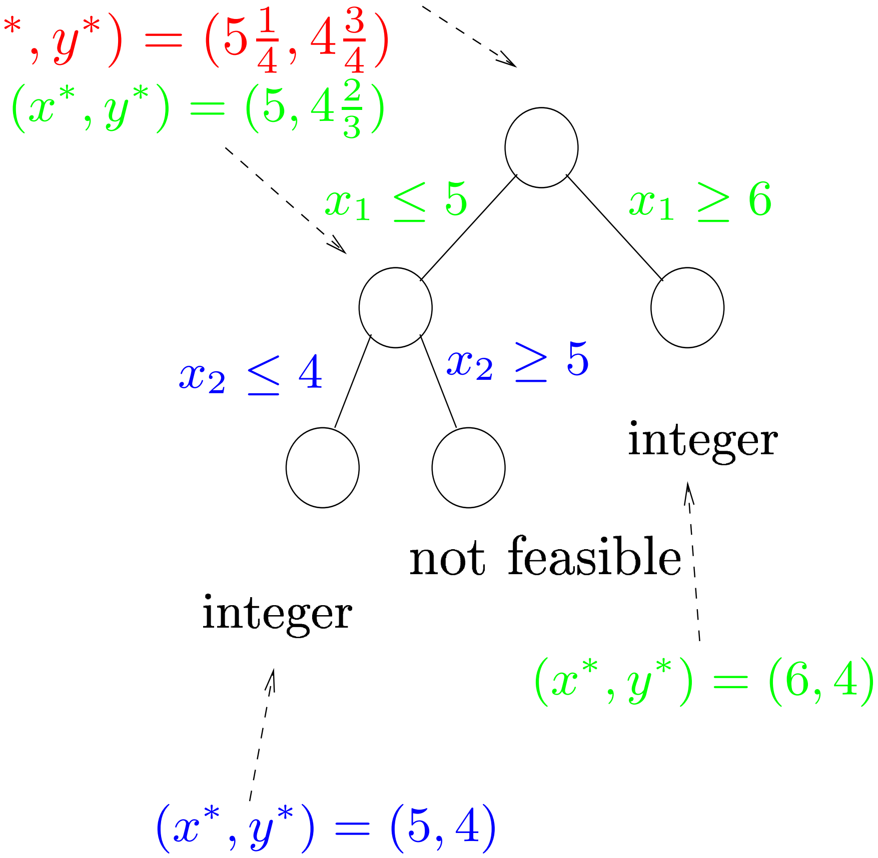
$$(x^*, y^*) = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

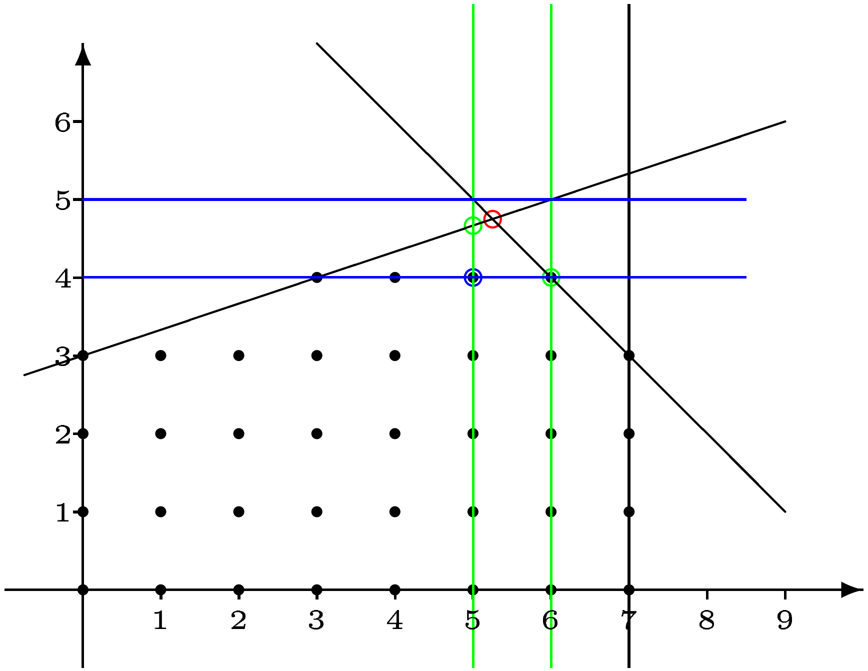
(The optimal extreme point to the continuous model is fractional)

Branch and bound (in e.g. Cplex)

Relax integrality requirements ⇒
linear, continuous problem ⇒ $(x^*, y^*) = (5\frac{1}{4}, 4\frac{3}{4})$

Search tree: branch over
fractional variable values





In the worst case ...

- It is reasonable to assume $T \approx 50$ time steps (or more)

$\Rightarrow$ 50 integer variables: $z_0, \ldots, z_{49}$

$\Rightarrow$ $2^{50} \approx 10^{15}$ branches

- Solve one continuous problem in 10^{-6} seconds $\Rightarrow$ 10^9seconds $\approx$ 30 years (10^{-9} seconds $\Rightarrow \approx$ 1.5 weeks)

- It is not really this bad for us, but:

  Better to generate facets so that all extreme points become integral

  A facet is a "best possible" cutting plane

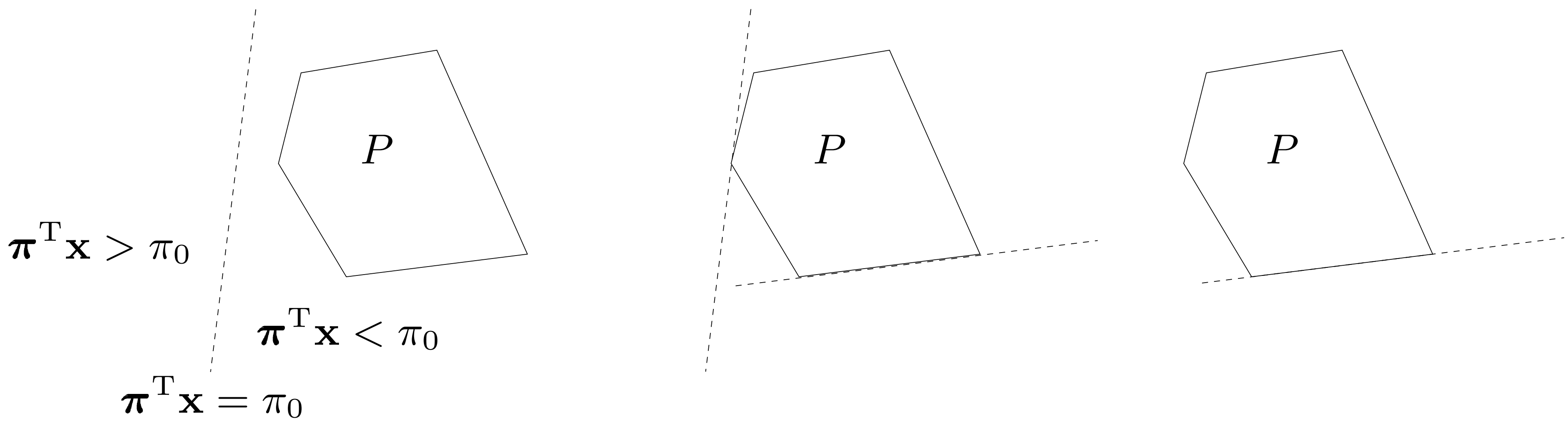
The smallest polyhedron containing all feasible points

- A general polyhedron defined by linear inequalities:
  $S = \left\{ \mathbf{x} \in \Re^n \,\middle|\, \mathbf{A}^{\mathrm{T}}\mathbf{x} \geq \mathbf{b} \right\}$
- The integer points of a (bounded) polyhedron defined by linear inequalities:
  $S_{\text{int}} = \left\{ \mathbf{x} \in \Re^n \,\middle|\, \mathbf{A}^{\mathrm{T}}\mathbf{x} \geq \mathbf{b}, \text{integral} \right\} = \left\{ \mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^K \right\}$
- We would like to find the convex hull of S_{int}: $P = \mathtt{conv} S_{\text{int}} =$
  $$\left\{ \mathbf{x} \in \Re^n \;\middle|\; \mathbf{x} = \sum_{k=1}^{K} \alpha_k \mathbf{x}^k, \quad \sum_{k=1}^{K} \alpha_k = 1, \quad \alpha_k \geq 0, k = 1, \ldots, K \right\}$$
- $P \subseteq S$ is also a polyhedron
- It then holds that $\displaystyle \min_{\mathbf{x} \in S_{\text{int}}} \mathbf{c}^{\mathrm{T}}\mathbf{x} = \min_{\mathbf{x} \in P} \mathbf{c}^{\mathrm{T}}\mathbf{x}$
- DRAW THE CONVEX HULL OF THE INTEGER POINTS!!

Valid inequalities, faces and facets

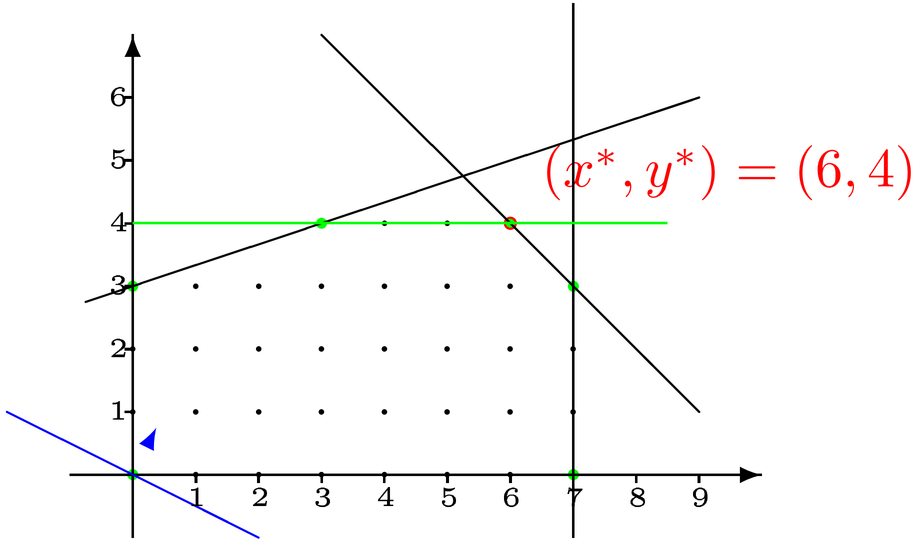
- $\boldsymbol{\pi}^{\mathrm{T}}\mathbf{x} \le \pi_0$ is a *valid inequality* for P if it holds for all $\mathbf{x} \in P$
- *Face*: $F = \left\{\mathbf{x} \in P \,\middle|\, \boldsymbol{\pi}^{\mathrm{T}}\mathbf{x} = \pi_0\right\}$ if $\boldsymbol{\pi}^{\mathrm{T}}\mathbf{x} \le \pi_0$ is a valid ineq. for P
- *Facet*: face of dimension $n-1$

Valid inequality for P | Faces of P | Facet of P



For the small example

- Find all facets $\Rightarrow$ no integrality requirements needed



A class of maintenance facets

In the basic problem, all necessary inequalities define facets!

For the components p and q such that the lives T_p and T_q fulfil

$$2 \leq T_q \leq T_p - 1 \leq 2 \cdot (T_q - 1)$$

a class of facets is defined by:

$$z_\ell + \sum_{t=\ell+1}^{\ell+T_p-2} (x_{pt} + x_{qt}) + z_{\ell+T_p-1} \geq 2, \qquad \ell = 1, \ldots, T - T_p + 1.$$

For the example ($T_1 = 3$, $T_2 = 5 \Rightarrow p = 2$, $q = 1$):

$$2 \leq T_1 = 3 \leq T_2 - 1 = 4 \leq 2 \cdot (T_1 - 1) = 4$$

A facet is given by the inequality

$$z_2 + x_{13} + x_{14} + x_{15} + x_{23} + x_{24} + x_{25} + z_6 \geq 2$$

Construction of a valid inequality

$$
\begin{aligned}
x_{12}+x_{13}+x_{14} \qquad\qquad\qquad\qquad\qquad\qquad &\geq 1 \\
x_{14}+x_{15}+x_{16} \qquad\qquad\qquad\qquad &\geq 1 \\
x_{22}+x_{23}+x_{24}+x_{25}+x_{26} &\geq 1
\end{aligned}
$$

Aggregate $\Rightarrow$ $x_{12}+x_{13}+2x_{14}+x_{15}+x_{16}+x_{22}+x_{23}+x_{24}+x_{25}+x_{26} \geq 3$ (1)

$$
\begin{aligned}
z_2 &\geq x_{12} \\
z_6 &\geq x_{16} \\
z_2 &\geq x_{22} \\
z_6 &\geq x_{26}
\end{aligned}
$$

Aggregate $\Rightarrow$ $2z_2 + 2z_6 \geq x_{12} + x_{16} + x_{22} + x_{26}$ (2)

(1) and (2) $\Rightarrow$ $2z_2 + x_{13} + 2x_{14} + x_{15} + x_{23} + x_{24} + x_{25} + 2z_6 \geq 3$ (3)

Construction cont'd

Multiply (3) by $\frac{1}{2}$:

$$z_2 + x_{13} + x_{14} + \frac{1}{2}x_{15} + \frac{1}{2}x_{23} + \frac{1}{2}x_{24} + \frac{1}{2}x_{25} + z_6 \geq \frac{3}{2} \tag{4}$$

Round-up the coefficients of the LHS to the nearest integer (OK?!):

$$z_2 + x_{13} + x_{14} + x_{15} + x_{23} + x_{24} + x_{25} + z_6 \geq \frac{3}{2}$$

All numbers in the LHS are integral in a feasible solution to the integer program $\Rightarrow$ Round-up the RHS to the nearest integer:

$$z_2 + x_{13} + x_{14} + x_{15} + x_{23} + x_{24} + x_{25} + z_6 \geq 2$$

We have shown that this inequality is *valid* for the *maintenance polytope*.

To show that it defines a *facet* takes a little more ...