MVE165/MMG630, Applied Optimization
Lecture 2

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Convex and concave functions

- A function f is convex on the set S if, for any elements $\mathbf{x}, \mathbf{y} \in S$ it holds that

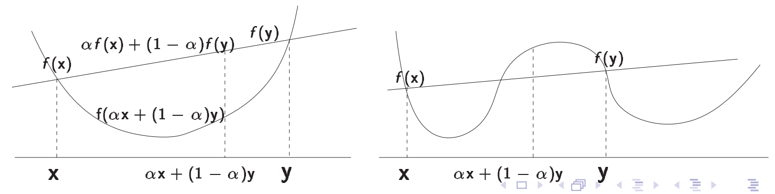
$$f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \leq \alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y}) \text{ for all } 0 \leq \alpha \leq 1$$

- A function f is concave on the set S if, for any elements $\mathbf{x}, \mathbf{y} \in S$ it holds that

$$f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \geq \alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y}) \text{ for all } 0 \leq \alpha \leq 1$$

$\Rightarrow$ Linear functions are convex (and concave)

Convex function　　　Non-convex function



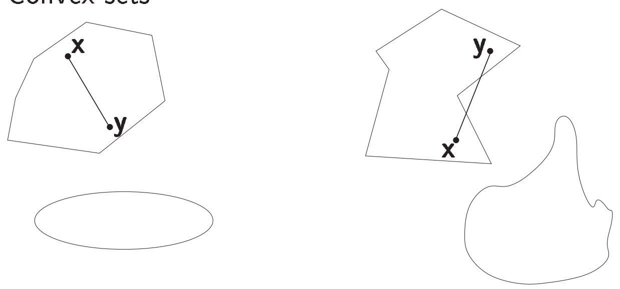
Convex sets

- A set S is convex if, for any elements $\mathbf{x}, \mathbf{y} \in S$ it holds that

$$\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \in S \text{ for all } 0 \leq \alpha \leq 1$$

- Examples:

Convex sets　　　Non-convex sets



$\Rightarrow$ Intersections of linear (in)equalities $\Rightarrow$ convex sets

Global solutions of convex programs

- Let $\mathbf{x}^*$ be a *local* minimizer of a *convex function* over a *convex set*. Then $\mathbf{x}^*$ is also a *global* minimizer.

$\Rightarrow$ Every local optimum of a linear program is a global optimum

- If a linear program has any optimal solutions, at least one optimal solution is at an extreme point of the feasible set

$\Rightarrow$ Search for optimal extreme point(s)

A general linear program

$$\text{minimize or maximize} \quad c_1x_1 + \ldots + c_nx_n$$

$$\text{subject to} \quad a_{i1}x_1 + \ldots + a_{in}x_n \left\{ \begin{matrix} \le \\ = \\ \ge \end{matrix} \right\} b_i, \quad i = 1, \ldots, m$$

$$x_j \left\{ \begin{matrix} \le 0 \\ \text{unrestricted in sign} \\ \ge 0 \end{matrix} \right\}, \quad j = 1, \ldots, n$$

- c_j, a_{ij}, and b_i are constant parameters for $i = 1, \ldots, m$ and $j = 1, \ldots, n$

The simplex method—reformulations

- The lego example:

$$\begin{bmatrix} 2x_1 & +x_2 & \le & 6 \\ 2x_1 & +2x_2 & \le & 8 \\ & x_1, x_2 & \ge & 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 2x_1 & +x_2 & +s_1 & & = & 6 \\ 2x_1 & +2x_2 & & +s_2 & = & 8 \\ & & & x_1, x_2, s_1, s_2 & \ge & 0 \end{bmatrix}$$

- s_1 and s_2 are called *slack variables*—they "fill out" the (positive) distances between the left and right hand sides
- *Surplus variable* s_3:

$$\begin{bmatrix} x_1 & + & x_2 & \ge & 800 \\ & & x_1, x_2 & \ge & 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} x_1 & + & x_2 & - & s_3 & = & 800 \\ & & & & x_1, x_2, s_3 & \ge & 0 \end{bmatrix}$$

The standard form and the simplex method for linear programs

- Every linear program can be reformulated such that:
  - all constraints are expressed as equalities with non-negative right hand sides
  - all variables are restricted to be non-negative
- Referred to as the *standard form*
- These requirements streamline the simplex method calculations
- Commercial solvers can handle also inequality constraints and unrestricted variables—the reformulations are automatically taken care of

The simplex method—reformulations, cont.

- Non-negative right hand side:

$$\begin{bmatrix} x_1 - x_2 & \le -23 \\ x_1, x_2 & \ge 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} -x_1 + x_2 & \ge 23 \\ x_1, x_2 & \ge 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} -x_1 + x_2 - s_4 & = 23 \\ x_1, x_2, s_4 & \ge 0 \end{bmatrix}$$

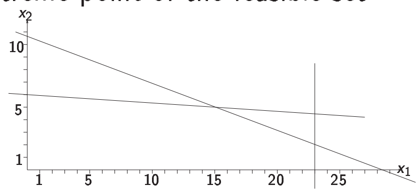
- Unrestricted variables:

$$\begin{bmatrix} x_1 + x_2 & \le 10 \\ x_1 & \ge 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} x_1 + x_2^1 - x_2^2 & \le 10 \\ x_1, x_2^1, x_2^2 & \ge 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} x_1 + x_2^1 - x_2^2 + s_5 & = 10 \\ x_1, x_2^1, x_2^2, s_5 & \ge 0 \end{bmatrix}$$

Basic feasible solutions

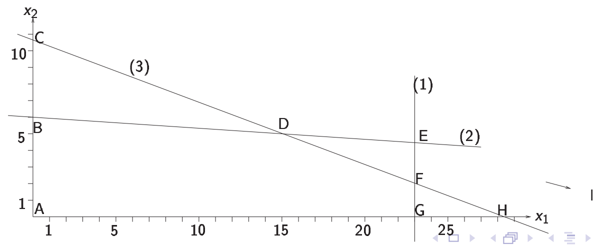
- Consider m equations of n variables, where $m \le n$
- Set $n - m$ variables to zero and solve (if possible) the remaining ($m \times m$) system of equations
- If the solution is *unique*, it is called a *basic* solution
- Such a solution corresponds to an intersection (feasible or infeasible) of m hyperplanes in $\Re^m$
- Each extreme point of the feasible set is an intersection of m hyperplanes with all variable values ≥ 0
- Basic feasible solution $\Leftrightarrow$ extreme point of the feasible set

$$\begin{aligned} a_{11}x_1 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \ldots + a_{2n}x_n &= b_2 \\ &\ldots \\ a_{m1}x_1 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$



Basic and non-basic variables and solutions

basic variables	basic solution			non-basic variables (0, 0)	point	feasible?
s_1, s_2, s_3	23	6	85	x_1, x_2	A	yes
s_1, s_2, x_1	$-5\frac{1}{3}$	$4\frac{1}{9}$	$28\frac{1}{3}$	s_3, x_2	H	no
s_1, s_2, x_2	23	$-4\frac{5}{8}$	$10\frac{5}{8}$	x_1, s_3	C	no
s_1, x_1, s_3	-67	90	-185	s_2, x_2	I	no
s_1, x_2, s_3	23	6	37	s_2, x_1	B	yes
x_1, s_2, s_3	23	$4\frac{7}{15}$	16	s_1, x_2	G	yes
x_2, s_2, s_3	-	-	-	s_1, x_1	-	-
x_1, x_2, s_1	15	5	8	s_2, s_3	D	yes
x_1, x_2, s_2	23	2	$2\frac{7}{15}$	s_1, s_3	F	yes
x_1, x_2, s_3	23	$4\frac{7}{15}$	$-19\frac{11}{15}$	s_1, s_2	E	no



Basic feasible solutions, example

- Constraints:

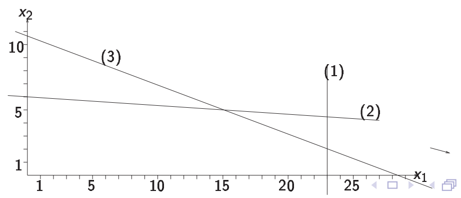
$$\begin{array}{rcrcl} x_1 & & & \le & 23 \quad (1) \\ 0.067x_1 & + & x_2 & \le & 6 \quad (2) \\ 3x_1 & + & 8x_2 & \le & 85 \quad (3) \\ & & x_1, x_2 & \ge & 0 \end{array}$$

- Add slack variables:

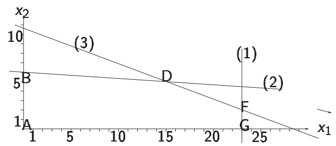
$$\begin{array}{rrrrrl} x_1 & & +s_1 & & & = 23 \quad (1) \\ 0.067x_1 & +x_2 & & +s_2 & & = 6 \quad (2) \\ 3x_1 & +8x_2 & & & +s_3 & = 85 \quad (3) \\ & & & & x_1, x_2, s_1, s_2, s_3 & \ge 0 \end{array}$$

$m = 3$

$n = 5$



Basic feasible solutions correspond to solutions to the system of equations that fulfil non-negativity



$$\begin{bmatrix} x_1 & & +s_1 & & & = 23 \\ 0.067x_1 & +x_2 & & +s_2 & & = 6 \\ 3x_1 & +8x_2 & & & +s_3 & = 85 \end{bmatrix}$$

A: $x_1 = x_2 = 0 \Rightarrow \begin{bmatrix} s_1 & & & = 23 \\ & s_2 & & = 6 \\ & & s_3 & = 85 \end{bmatrix}$

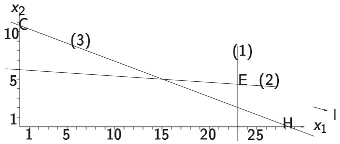
B: $x_1 = s_2 = 0 \Rightarrow \begin{bmatrix} & s_1 & & = 23 \\ x_2 & & & = 6 \\ 8x_2 & & +s_3 & = 85 \end{bmatrix}$

D: $s_3 = s_2 = 0 \Rightarrow \begin{bmatrix} x_1 & & +s_1 & = 23 \\ 0.067x_1 & +x_2 & & = 6 \\ 3x_1 & +8x_2 & & = 85 \end{bmatrix}$

F: $s_3 = s_1 = 0 \Rightarrow \begin{bmatrix} x_1 & & & = 23 \\ 0.067x_1 & +x_2 & +s_2 & = 6 \\ 3x_1 & +8x_2 & & = 85 \end{bmatrix}$

G: $x_2 = s_1 = 0 \Rightarrow \begin{bmatrix} x_1 & & & = 23 \\ 0.067x_1 & +s_2 & & = 6 \\ 3x_1 & & +s_3 & = 85 \end{bmatrix}$

Basic **infeasible** solutions correspond to solutions to the system of equations with one or more variables < 0



$$\begin{bmatrix} x_1 & & +s_1 & & & = 23 \\ 0.067x_1 & +x_2 & & +s_2 & & = 6 \\ 3x_1 & +8x_2 & & & +s_3 & = 85 \end{bmatrix}$$

H: $x_2 = s_3 = 0 \Rightarrow \begin{bmatrix} x_1 & +s_1 & & = 23 \\ 0.067x_1 & & +s_2 & = 6 \\ 3x_1 & & & = 85 \end{bmatrix}$

C: $x_1 = s_3 = 0 \Rightarrow \begin{bmatrix} & s_1 & & = 23 \\ x_2 & & +s_2 & = 6 \\ 8x_2 & & & = 85 \end{bmatrix}$

I: $s_2 = x_2 = 0 \Rightarrow \begin{bmatrix} x_1 & +s_1 & & = 23 \\ 0.067x_1 & & & = 6 \\ 3x_1 & & +s_3 & = 85 \end{bmatrix}$

-: $s_1 = x_1 = 0 \Rightarrow \begin{bmatrix} & & 0 & = 23 \\ x_2 & +s_2 & & = 6 \\ 8x_2 & & +s_3 & = 85 \end{bmatrix}$

E: $s_1 = s_2 = 0 \Rightarrow \begin{bmatrix} x_1 & & & = 23 \\ 0.067x_1 & +x_2 & & = 6 \\ 3x_1 & +8x_2 & +s_3 & = 85 \end{bmatrix}$

Basic feasible solutions and the simplex method

- The *first basic solution* can be represented as

$-z$	$+2x_1$	$+3x_2$				$= 0$	(0)
	x_1		$+s_1$			$= 23$	(1)
	$\frac{1}{15}x_1$	$+x_2$		$+s_2$		$= 6$	(2)
	$3x_1$	$+8x_2$			$+s_3$	$= 85$	(3)

- Marginal values for increasing the non-basic variables x_1 and x_2 from zero: 2 and 3, resp.

$\Rightarrow$ Choose x_2 — let x_2 *enter the basis* DRAW GRAPH!!

- One basic variable (s_1, s_2, or s_3) must leave the basis. Which?
- The value of x_2 can increase until some basic variable reaches the value 0:

$$\left.\begin{array}{ll} (2): s_2 = 6 - x_2 \geq 0 & \Rightarrow x_2 \leq 6 \\ (3): s_3 = 85 - 8x_2 \geq 0 & \Rightarrow x_2 \leq 10\frac{5}{8} \end{array}\right\} \Rightarrow \begin{array}{c} s_2 = 0 \text{ when} \\ x_2 = 6 \\ (\text{and } s_3 = 37) \end{array}$$

- s_2 will leave the basis

Basic feasible solutions and the simplex method

- Express the m *basic* variables in terms of the $n - m$ *non-basic* variables
- Example: Start at $x_1 = x_2 = 0 \Rightarrow s_1, s_2, s_3$ are *basic*

$$\begin{bmatrix} x_1 & & +s_1 & & & = 23 \\ \frac{1}{15}x_1 & +x_2 & & +s_2 & & = 6 \\ 3x_1 & +8x_2 & & & +s_3 & = 85 \end{bmatrix}$$

- Express s_1, s_2, and s_3 in terms of x_1 and x_2:

$$\begin{bmatrix} s_1 = & 23 & -x_1 & \\ s_2 = & 6 & -\frac{1}{15}x_1 & -x_2 \\ s_3 = & 85 & -3x_1 & -8x_2 \end{bmatrix}$$

- Express the objective in terms of the *non-basic* variables:

$$z = 2x_1 + 3x_2 \quad \Leftrightarrow \quad z - 2x_1 - 3x_2 = 0$$

Change basis through row operations

- Eliminate s_2 from the basis, let x_2 enter the basis using row operations:

$-z$	$+2x_1$	$+3x_2$				$=$	0	(0)
	x_1		$+s_1$			$=$	23	(1)
	$\frac{1}{15}x_1$	$+\mathbf{x_2}$		$+s_2$		$=$	6	(2)
	$3x_1$	$+8x_2$			$+s_3$	$=$	85	(3)
$-z$	$+\frac{9}{5}x_1$			$-3s_2$		$=$	-18	(0) $-3\cdot$(2)
	x_1		$+s_1$			$=$	23	(1)$-0\cdot$(2)
	$\frac{1}{15}x_1$	$+x_2$		$+s_2$		$=$	6	(2)
	$\frac{37}{15}x_1$			$-8s_2$	$+s_3$	$=$	37	(3)$-8\cdot$(2)

- Corresponding basic solution: $s_1 = 23$, $x_2 = 6$, $s_3 = 37$.
- Nonbasic variables: $x_1 = s_2 = 0$
- The marginal value of x_1 is $\frac{9}{5} > 0$. Let x_1 enter the basis
- Which should leave? s_1, x_2, or s_3?

Change basis ...

$$
\begin{array}{lllllll|r|l}
-z & +\frac{9}{5}x_1 & & & -3s_2 & & = & -18 & (0)\\
 & x_1 & & +s_1 & & & = & 23 & (1)\\
 & \frac{1}{15}x_1 & +x_2 & & +s_2 & & = & 6 & (2)\\
 & \mathbf{\frac{37}{15}x_1} & & & -8s_2 & +s_3 & = & 37 & (3)
\end{array}
$$

- The value of x_1 can increase until some basic variable reaches the value 0:

$$
\left.\begin{array}{ll}
(1): s_1 = 23 - x_1 \geq 0 & \Rightarrow x_1 \leq 23\\
(2): x_2 = 6 - \frac{1}{15}x_1 \geq 0 & \Rightarrow x_1 \leq 90\\
(3): s_3 = 37 - \frac{37}{15}x_1 \geq 0 & \Rightarrow x_1 \leq 15
\end{array}\right\} \Rightarrow \begin{array}{l} s_3 = 0 \text{ when}\\ x_1 = 15\end{array}
$$

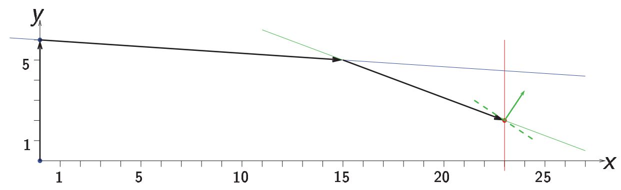
- x_1 enters the basis and s_3 will leave the basis
- Perform row operations:

$$
\begin{array}{lllll|r|l}
-z & & & +2.84s_2 & -0.73s_3 & = -45 & (0)-(3)\cdot\frac{15}{37}\cdot\frac{9}{5}\\
 & & s_1 & +3.24s_2 & -0.41s_3 & = 8 & (1)-(3)\cdot\frac{15}{37}\\
 & x_2 & & +1.22s_2 & -0.03s_3 & = 5 & (2)-(3)\cdot\frac{15}{37}\cdot\frac{1}{15}\\
x_1 & & & -3.24s_2 & +0.41s_3 & = 15 & (3)\cdot\frac{15}{37}
\end{array}
$$

Optimal basic solution

$$
\begin{array}{llllll|r}
-z & & -0.87s_1 & & -0.37s_3 & = & -52\\
 & & 0.31s_1 & +s_2 & -0.12s_3 & = & 2.47\\
 & x_2 & -0.37s_1 & & +0.12s_3 & = & 2\\
x_1 & & +s_1 & & & = & 23
\end{array}
$$

- No marginal value is positive. No improvement can be made
- The optimal basis is given by $s_2 = 2.47$, $x_2 = 2$, and $x_1 = 23$
- Non-basic variables: $s_1 = s_3 = 0$
- Optimal value: $z = 52$



Change basis ...

$$
\begin{array}{lllll|r|l}
-z & & & +2.84s_2 & -0.73s_3 & = -45 & (0)\\
 & & s_1 & \mathbf{+3.24s_2} & -0.41s_3 & = 8 & (1)\\
 & x_2 & & +1.22s_2 & -0.03s_3 & = 5 & (2)\\
x_1 & & & -3.24s_2 & +0.41s_3 & = 15 & (3)
\end{array}
$$

- Let s_2 enter the basis (marginal value > 0)
- The value of s_2 can increase until some basic variable $= 0$:

$$
\left.\begin{array}{ll}
(1): s_1 = 8 - 3.24s_2 \geq 0 & \Rightarrow s_2 \leq 2.47\\
(2): x_2 = 5 - 1.22s_2 \geq 0 & \Rightarrow s_2 \leq 4.10\\
(3): x_1 = 15 + 3.24s_2 \geq 0 & \Rightarrow s_2 \geq -4.63
\end{array}\right\} \Rightarrow \begin{array}{l} s_1 = 0 \text{ when}\\ s_2 = 2.47\end{array}
$$

- s_2 enters the basis and s_1 will leave the basis
- Perform row operations:

$$
\begin{array}{llllll|r|l}
-z & & -0.87s_1 & & -0.37s_3 & = & -52 & (0)-(1)\cdot\frac{2.84}{3.24}\\
 & & 0.31s_1 & +s_2 & -0.12s_3 & = & 2.47 & (1)\cdot\frac{1}{3.24}\\
 & x_2 & -0.37s_1 & & +0.12s_3 & = & 2 & (2)-(1)\cdot\frac{1.22}{3.24}\\
x_1 & & +s_1 & & & = & 23 & (3)+(1)
\end{array}
$$

Summary of the solution course

basis	$-z$	x_1	x_2	s_1	s_2	s_3	RHS
$-z$	1	2	3	0	0	0	0
s_1	0	1	0	1	0	0	23
s_2	0	0.067	**1**	0	1	0	6
s_3	0	3	8	0	0	1	85
$-z$	1	1.80	0	0	-3	0	-18
s_1	0	1	0	1	0	0	23
x_2	0	0.07	1	0	1	0	6
s_3	0	**2.47**	0	0	-8	1	37
$-z$	1	0	0	0	2.84	-0.73	-45
s_1	0	0	0	1	**3.24**	-0.41	8
x_2	0	0	1	0	1.22	-0.03	5
x_1	0	1	0	0	-3.24	0.41	15
$-z$	1	0	0	-0.87	0	-0.37	-52
s_2	0	0	0	0.31	1	-0.12	2.47
x_2	0	0	1	-0.37	0	0.12	2
x_1	0	1	0	1	0	0	23

Summary of the simplex method

- **Optimality condition**: The *entering* variable in a maximization (minimization) problem should have the largest positive (negative) marginal value (reduced cost).

  The entering variable determines a direction in which the objective value increases (decreases).

  If all reduced costs are negative (positive), the current basis is optimal.

- **Feasibility condition**: The *leaving* variable is the one with smallest nonnegative ratio.

  Corresponds to the constraint that is "reached first"

Solve the lego problem using the simplex method!

$$\begin{array}{llrcrcl} \text{maximize} & z = & 1600x_1 & + & 1000x_2 & & \\ \text{subject to} & & 2x_1 & + & x_2 & \leq & 6 \\ & & 2x_1 & + & 2x_2 & \leq & 8 \\ & & & & x_1,\ x_2 & \geq & 0 \end{array}$$

HOMEWORK!!

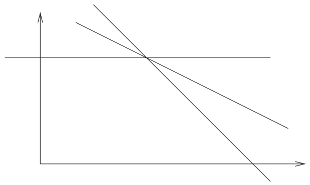
Simplex search for linear (minimization) programs (p. 204)

1. **Initialization:** Choose any feasible basis, construct the corresponding basic solution $\mathbf{x}^0$, let $t = 0$
2. **Step direction:** Select a variable to enter the basis using the optimality condition (negative marginal value). Stop if no entering variable exists
3. **Step length:** Select a leaving variable using the feasibility condition (smallest non-negative ratio)
4. **New iterate:** Compute the new basic solution $\mathbf{x}^{t+1}$ by performing matrix operations.
5. Let $t := t + 1$ and repeat from 2

Degeneracy (Ch. 5.6)

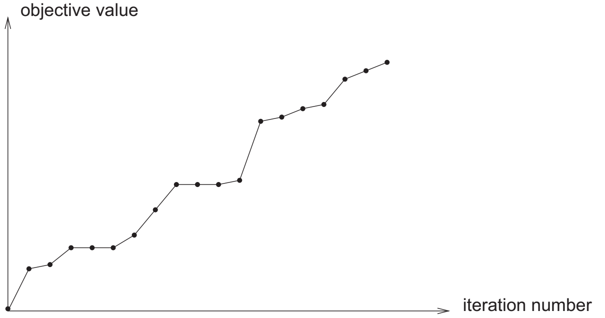
- If the smallest nonnegative ratio is zero, the value of a basic variable will become zero in the next iteration
- The solution is *degenerate*
- The objective value will *not* improve in this iteration
- Risk: *cycling* around (non-optimal) bases
- Reason: a *redundant* constraint "touches" the feasible set
- Example:

$$\begin{array}{rcrcl} x_1 & + & x_2 & \leq & 6 \\ & & x_2 & \leq & 3 \\ x_1 & + & 2x_2 & \leq & 9 \\ & & x_1,\ x_2 & \geq & 0 \end{array}$$



Degeneracy

- Typical objective function progress of the simplex method



- Computational rules to prevent from infinite cycling: careful choices of leaving and entering variables
- In modern software: perturb the right hand side $(b_i + \Delta b_i)$, solve, reduce the perturbation and resolve from the current basis. Repeat until $\Delta b_i = 0$.

Unbounded solutions

- A feasible basis is given by $x_1 = 1$, $x_2 = 3$, with corresponding tableau:
  Homework: Find this basis using the simplex method.

basis	$-z$	x_1	x_2	s_1	s_2	RHS
$-z$	1	0	0	5	-3	7
x_1	0	1	0	1	-1	1
x_2	0	0	1	2	-1	3

- Entering variable is s_2
- Row 1: $x_1 = 1 + s_2 \geq 0 \Rightarrow s_2 \geq -1$
- Row 2: $x_2 = 3 + s_2 \geq 0 \Rightarrow s_2 \geq -3$
- No leaving variable can be found, since no constraint will prevent s_2 from increasing infinitely

Unbounded solutions

- If all ratios are negative, the variable entering the basis may increase infinitely
- The feasible set is unbounded
- In a real application this would probably be due to some incorrect assumption
- Example:

$$\begin{array}{lrrrl} \text{minimize} & z = & -x_1 & -2x_2 & \\ \text{subject to} & & -x_1 & +x_2 & \leq 2 \\ & & -2x_1 & +x_2 & \leq 1 \\ & & & x_1,\ x_2 & \geq 0 \end{array}$$

DRAW GRAPH!!

Starting solution—finding an initial basis (Ch. 5.5, p. 211)

- Example:

$$\begin{array}{lrrrl} \text{minimize} & z = & 2x_1 & +3x_2 & \\ \text{subject to} & & 3x_1 & +2x_2 & = 14 \\ & & 2x_1 & -4x_2 & \geq 2 \\ & & 4x_1 & +3x_2 & \leq 19 \\ & & & x_1, x_2 & \geq 0 \end{array}$$

DRAW GRAPH!!

- Add slack and surplus variables

$$\begin{array}{lrrrrrl} \text{minimize} & z = & 2x_1 & +3x_2 & & & \\ \text{subject to} & & 3x_1 & +2x_2 & & & = 14 \\ & & 2x_1 & -4x_2 & -s_1 & & = 2 \\ & & 4x_1 & +3x_2 & & +s_2 & = 19 \\ & & & & & x_1, x_2, s_1, s_2 & \geq 0 \end{array}$$

- How finding an initial basis? Only s_2 is obvious!

Artificial variables

- Add artificial variables a_1 and a_2 to the first and second constraints, respectively
- Solve an artificial problem: minimize $a_1 + a_2$

$$
\begin{array}{llrrrrrrl}
\text{minimize} & w = & & & & & a_1 & +a_2 & \\
\text{subject to} & & 3x_1 & +2x_2 & & & +a_1 & & = 14 \\
& & 2x_1 & -4x_2 & -s_1 & & & +a_2 & = 2 \\
& & 4x_1 & +3x_2 & & +s_2 & & & = 19 \\
& & & & & & \multicolumn{2}{r}{x_1, x_2, s_1, s_2, a_1, a_2} & \geq 0
\end{array}
$$

- The "phase one" problem
- An initial basis is given by $a_1 = 14$, $a_2 = 2$, and $s_2 = 19$:

basis	$-w$	x_1	x_2	s_1	s_2	a_1	a_2	RHS
$-w$	1	-5	2	1	0	0	0	-16
a_1	0	3	2	0	0	1	0	14
a_2	0	2	-4	-1	0	0	1	2
s_2	0	4	3	0	1	0	0	19

Infeasible linear programs

- If the solution to the "phase one" problem has optimal value $= 0$, a feasible basis has been found

$\Rightarrow$ Start optimizing the original objective function z from this basis (*homework*)

- If the solution to the "phase one" problem has optimal value $w > 0$, no feasible solutions exist
- What would this mean in a real application?
- Alternative: Big-M method: Add the artificial variables to the original objective—with a large coefficient
  Example:

$$\text{minimize} \quad z = 2x_1 + 3x_2$$

$$\Rightarrow \quad \text{minimize} \quad z_a = 2x_1 + 3x_2 + Ma_1 + Ma_2$$

Find an initial solution using artificial variables

- x_1 enters $\Rightarrow$ a_2 leaves (then $x_2 \Rightarrow s_2$, then $s_1 \Rightarrow a_1$)

basis	$-w$	x_1	x_2	s_1	s_2	a_1	a_2	RHS
$-w$	1	-5	2	1	0	0	0	-16
a_1	0	3	2	0	0	1	0	14
a_2	0	2	-4	-1	0	0	1	2
s_2	0	4	3	0	1	0	0	19
$-w$	1	0	-8	-1.5	0	0		-11
a_1	0	0	8	1.5	0	1		11
x_1	0	1	-2	-0.5	0	0		1
s_2	0	0	11	2	1	0		15
$-w$	1	0	0	-0.045	0.727	0		-0.091
a_1	0	0	0	0.045	-0.727	1		0.091
x_1	0	1	0	-0.136	0.182	0		3.727
x_2	0	0	1	0.182	0.091	0		1.364
$-w$	1	0	0	0	0			0
s_1	0	0	0	1	-16			2
x_1	0	1	0	0	-2			4
x_2	0	0	1	0	3			1

- A feasible basis is given by $x_1 = 4$, $x_2 = 1$, and $s_1 = 2$

Alternative optimal solutions

- Example:

$$
\begin{array}{llrrl}
\text{minimize} & z = & 2x_1 & +4x_2 & \\
\text{subject to} & & x_1 & +2x_2 & \leq 5 \\
& & x_1 & +x_2 & \leq 4 \\
& & & x_1, x_2 & \geq 0
\end{array}
$$

DRAW GRAPH!!

- The extreme points $(0, \frac{5}{2})$ and $(3, 1)$ have the same optimal value $z = 10$
- All solutions that are positive linear (convex) combinations of these are optimal:

$$(x_1, x_2) = \alpha \cdot (0, \frac{5}{2}) + (1 - \alpha) \cdot (3, 1), \quad 0 \leq \alpha \leq 1$$