

# MVE165/MMG630, Applied Optimization

## Lecture 3

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2009-03-20

Lecture 3 Applied Optimization

### Alternative optimal solutions

- ▶ Example:

$$\begin{aligned} \text{maximize } z &= 2x_1 + 4x_2 \\ \text{subject to } x_1 + 2x_2 &\leq 5 \\ x_1 + x_2 &\leq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

DRAW GRAPH!!

- ▶ The extreme points  $(0, \frac{5}{2})$  and  $(3, 1)$  have the same optimal value  $z = 10$
- ▶ All solutions that are positive linear (convex) combinations of these are optimal:

$$(x_1, x_2) = \alpha \cdot (0, \frac{5}{2}) + (1 - \alpha) \cdot (3, 1), \quad 0 \leq \alpha \leq 1$$

- ▶ Reduced cost of a non-basic variable is 0 in an optimal basis

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### A general linear program in standard form

- ▶ A linear program with  $n$  non-negative variables,  $m$  equality constraints ( $m < n$ ), and non-negative right hand sides:

$$\begin{aligned} \text{maximize } z &= \sum_{j=1}^n c_j x_j \\ \text{subject to } \sum_{j=1}^n a_{ij} x_j &= b_i, \quad i = 1, \dots, m, \\ x_j &\geq 0, \quad j = 1, \dots, n. \end{aligned}$$

- ▶ On matrix form it is written as:

$$\begin{aligned} \text{maximize } z &= \mathbf{c}^T \mathbf{x}, \\ \text{subject to } \mathbf{A} \mathbf{x} &= \mathbf{b}, \\ \mathbf{x} &\geq \mathbf{0}^n, \end{aligned}$$

where  $\mathbf{x} \in \mathbb{R}^n$ ,  $\mathbf{A} \in \mathbb{R}^{m \times n}$ ,  $\mathbf{b} \in \mathbb{R}_+^m$  ( $\mathbf{b} \geq \mathbf{0}^m$ ), and  $\mathbf{c} \in \mathbb{R}^n$ .

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### General derivation of the simplex method

- ▶  $B$  = set of basic variables,  $N$  = set of non-basic variables
- ⇒  $|B| = m$  and  $|N| = n - m$
- ▶ Partition matrix/vectors:  $\mathbf{A} = (\mathbf{B}, \mathbf{N})$ ,  $\mathbf{x} = (\mathbf{x}_B, \mathbf{x}_N)$ ,  $\mathbf{c} = (\mathbf{c}_B, \mathbf{c}_N)$
- ▶ The matrix  $\mathbf{B}$  ( $\mathbf{N}$ ) contains the columns of  $\mathbf{A}$  corresponding to the index set  $B$  ( $N$ ) — Analogously for  $\mathbf{x}$  and  $\mathbf{c}$
- ▶ Rewrite the linear program:

$$\left[ \begin{array}{l} \text{maximize } z = \mathbf{c}^T \mathbf{x} \\ \text{subject to } \mathbf{A} \mathbf{x} = \mathbf{b}, \\ \mathbf{x} \geq \mathbf{0}^n \end{array} \right] = \left[ \begin{array}{l} \text{maximize } z = \mathbf{c}_B^T \mathbf{x}_B + \mathbf{c}_N^T \mathbf{x}_N \\ \text{subject to } \mathbf{B} \mathbf{x}_B + \mathbf{N} \mathbf{x}_N = \mathbf{b}, \\ \mathbf{x}_B \geq \mathbf{0}^m, \mathbf{x}_N \geq \mathbf{0}^{n-m} \end{array} \right]$$

- ▶ Substitute:  $\mathbf{x}_B = \mathbf{B}^{-1} \mathbf{b} - \mathbf{B}^{-1} \mathbf{N} \mathbf{x}_N \Rightarrow$

$$\begin{aligned} \text{maximize } z &= \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{b} + [\mathbf{c}_N^T - \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{N}] \mathbf{x}_N \\ \text{subject to } \mathbf{B}^{-1} \mathbf{b} - \mathbf{B}^{-1} \mathbf{N} \mathbf{x}_N &\geq \mathbf{0}^m, \\ \mathbf{x}_N &\geq \mathbf{0}^{n-m} \end{aligned}$$

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## Optimality and feasibility

### ▶ Optimality condition (for maximization)

The basis  $B$  is optimal if  $\mathbf{c}_N^T - \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{N} \leq \mathbf{0}^{n-m}$   
(marginal values = reduced costs  $\leq 0$ )

If not, choose as entering variable  $j \in N$  the one with the largest value of the reduced cost  $c_j - \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{A}_j$

### ▶ Feasibility condition

For all  $i \in B$  it holds that  $x_i = (\mathbf{B}^{-1} \mathbf{b})_i - (\mathbf{B}^{-1} \mathbf{A}_j)_i x_j$

Choose the leaving variable  $i^* \in B$  according to

$$i^* = \arg \min_{i \in B} \left\{ \frac{(\mathbf{B}^{-1} \mathbf{b})_i}{(\mathbf{B}^{-1} \mathbf{A}_j)_i} \mid (\mathbf{B}^{-1} \mathbf{A}_j)_i > 0 \right\}$$

## In the simplex tableau we have

basis	$-z$	$\mathbf{x}_B$	$\mathbf{x}_N$	$\mathbf{s}$	RHS
$-z$	1	$\mathbf{0}$	$\mathbf{c}_N^T - \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{N}$	$-\mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{s}$	$-\mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{b}$
$\mathbf{x}_B$	$\mathbf{0}$	$\mathbf{I}$	$\mathbf{B}^{-1} \mathbf{N}$	$\mathbf{B}^{-1}$	$\mathbf{B}^{-1} \mathbf{b}$

- ▶  $\mathbf{s}$  denotes possible slack variables (columns for  $\mathbf{s}$  are copies of certain columns for  $(\mathbf{x}_B, \mathbf{x}_N)$ )
  - ▶ The computations performed by the simplex algorithm involve matrix inversions and updates of these
  - ▶ A non-basic (basic) variable enters (leaves) the basis  $\Rightarrow$  one column,  $\mathbf{A}_j$ , of  $\mathbf{B}$  is replaced by another,  $\mathbf{A}_k$
  - ▶ Row operations  $\Leftrightarrow$  Updates of  $\mathbf{B}^{-1}$  (and  $\mathbf{B}^{-1} \mathbf{N}$ ,  $\mathbf{B}^{-1} \mathbf{b}$ , and  $\mathbf{c}_B^T \mathbf{B}^{-1}$ )
- $\Rightarrow$  Efficient numerical computations are crucial for the performance of the simplex algorithm

## Derivation of duality

### ▶ A linear program with optimal value $z^*$

$$\begin{array}{ll} \text{maximize } z := & 20x_1 + 18x_2 \\ \text{subject to} & 7x_1 + 10x_2 \leq 3600 \quad (1) \\ & 16x_1 + 12x_2 \leq 5400 \quad (2) \\ & x_1, x_2 \geq 0 \end{array} \quad \begin{array}{l} \text{weights} \\ v_1 \\ v_2 \end{array}$$

### ▶ How large can $z^*$ be?

### ▶ Compute upper estimates of $z^*$ , e.g.

- ▶ Multiply (1) by 3  $\Rightarrow 21x_1 + 30x_2 \leq 10800 \Rightarrow z^* \leq 10800$
- ▶ Multiply (2) by 1.5  $\Rightarrow 24x_1 + 18x_2 \leq 8100 \Rightarrow z^* \leq 8100$
- ▶ Combine:  $0.6 \times (1) + 1 \times (2) \Rightarrow 20.2x_1 + 18x_2 \leq 7560 \Rightarrow z^* \leq 7560$

### ▶ Do better than guess—compute optimal weights!

### ▶ Value of estimate: $w = 3600v_1 + 5400v_2 \rightarrow \min$

$$\text{Constraints on weights: } \begin{cases} 7v_1 + 16v_2 \geq 20 \\ 10v_1 + 12v_2 \geq 18 \\ v_1, v_2 \geq 0 \end{cases}$$

## The best (lowest) possible upper estimate of $z^*$

$$\begin{array}{ll} \text{minimize } w := & 3600v_1 + 5400v_2 \\ \text{subject to} & 7v_1 + 16v_2 \geq 20 \\ & 10v_1 + 12v_2 \geq 18 \\ & v_1, v_2 \geq 0 \end{array}$$

### ▶ A linear program!

### ▶ It is called the **dual** of the original linear program

## The lego model—the market problem

- ▶ Consider the lego problem

$$\begin{aligned} \text{maximize } z &= 1600x_1 + 1000x_2 \\ \text{subject to } & 2x_1 + x_2 \leq 6 \\ & 2x_1 + 2x_2 \leq 8 \\ & x_1, x_2 \geq 0 \end{aligned}$$

- ▶ Option: Sell blocks instead of making furniture
- ▶  $v_1(v_2)$  = price of a large (small) block
- ▶ Market wish to minimize payment: *minimize*  $6v_1 + 8v_2$
- ▶ I sell if prices are high enough:
  - ▶  $2v_1 + 2v_2 \geq 1600$  – otherwise better to make tables
  - ▶  $v_1 + 2v_2 \geq 1000$  – otherwise better to make chairs
  - ▶  $v_1, v_2 \geq 0$  – prices are naturally non-negative

## In practice ...

- ▶ A primal linear program

$$\begin{aligned} \text{minimize } z &= 2x_1 + 3x_2 \\ \text{subject to } & 3x_1 + 2x_2 = 14 \\ & 2x_1 - 4x_2 \geq 2 \\ & 4x_1 + 3x_2 \leq 19 \\ & x_1, x_2 \geq 0 \end{aligned}$$

- ▶ The corresponding dual linear program

$$\begin{aligned} \text{maximize } w &= 14y_1 + 2y_2 + 19y_3 \\ \text{subject to } & 3y_1 + 2y_2 + 4y_3 \leq 2 \\ & 2y_1 - 4y_2 + 3y_3 \leq 3 \\ & y_1 \text{ free,} \\ & y_2 \geq 0, \\ & y_3 \leq 0 \end{aligned}$$

## Linear programming duality

- ▶ To each primal linear program corresponds a dual linear program

$$\begin{aligned} \text{[Primal] minimize } z &= \mathbf{c}^T \mathbf{x}, \\ \text{subject to } \mathbf{Ax} &= \mathbf{b}, \\ \mathbf{x} &\geq \mathbf{0}^n, \end{aligned}$$

$$\begin{aligned} \text{[Dual] maximize } w &= \mathbf{b}^T \mathbf{y}, \\ \text{subject to } \mathbf{A}^T \mathbf{y} &\leq \mathbf{c}. \end{aligned}$$

- ▶ On component form:
  - minimize  $z = \sum_{j=1}^n c_j x_j$
  - subject to  $\sum_{j=1}^n a_{ij} x_j = b_i, \quad i = 1, \dots, m,$
  - $x_j \geq 0, \quad j = 1, \dots, n,$
  - maximize  $w = \sum_{i=1}^m b_i y_i$
  - subject to  $\sum_{i=1}^m a_{ij} y_i \leq c_j, \quad j = 1, \dots, n.$

## Rules for constructing the dual program (p. 327)

maximization	$\Leftrightarrow$	minimization
dual program	$\Leftrightarrow$	primal program
primal program	$\Leftrightarrow$	dual program
<i>constraints</i>		<i>variables</i>
$\geq$	$\Leftrightarrow$	$\leq 0$
$\leq$	$\Leftrightarrow$	$\geq 0$
$=$	$\Leftrightarrow$	free
<i>variables</i>		<i>constraints</i>
$\geq 0$	$\Leftrightarrow$	$\geq$
$\leq 0$	$\Leftrightarrow$	$\leq$
free	$\Leftrightarrow$	$=$

The dual of the dual of any linear program equals the primal

## Duality properties (Ch. 7.5)

- ▶ **Weak duality:** Let  $\mathbf{x}$  be a feasible point in the primal and  $\mathbf{y}$  be a feasible point in the dual. Then,

$$z = \mathbf{c}^T \mathbf{x} \geq \mathbf{b}^T \mathbf{y} = w$$

- ▶ **Strong duality:** In a pair of primal and dual linear programs, if one of them has an optimal solution, so does the other, and their optimal values are equal.

- ▶ **Complementary slackness:** If  $\mathbf{x}$  is optimal in the primal and  $\mathbf{y}$  is optimal in the dual, then  $\mathbf{x}^T(\mathbf{c} - \mathbf{A}^T \mathbf{y}) = \mathbf{y}^T(\mathbf{b} - \mathbf{A} \mathbf{x}) = 0$ .

If  $\mathbf{x}$  is feasible in the primal,  $\mathbf{y}$  is feasible in the dual, and  $\mathbf{x}^T(\mathbf{c} - \mathbf{A}^T \mathbf{y}) = \mathbf{y}^T(\mathbf{b} - \mathbf{A} \mathbf{x}) = 0$ , then  $\mathbf{x}$  and  $\mathbf{y}$  are optimal for their respective problems.



## Exercises on duality

- ▶ Formulate and solve graphically the dual of:

$$\begin{aligned} \text{minimize } z &= 6x_1 + 3x_2 + x_3 \\ \text{subject to } &6x_1 - 3x_2 + x_3 \geq 2 \\ &3x_1 + 4x_2 + x_3 \geq 5 \\ &x_1, x_2, x_3 \geq 0 \end{aligned}$$

- ▶ Then find the optimal primal solution
- ▶ Verify that the dual of the dual equals the primal



## Relations between primal and dual optimal solutions

primal (dual) problem	$\iff$	dual (primal) problem
unique and non-degenerate solution	$\iff$	unique and non-degenerate solution
unbounded solution	$\implies$	no feasible solutions
no feasible solutions	$\implies$	unbounded solution <b>or</b> no feasible solutions
degenerate solution	$\iff$	alternative solutions



## Sensitivity analysis

- ▶ How does the optimum change when the right hand sides (resources, e.g.) change?
- ▶ When the objective coefficients (prices, e.g.) change?
- ▶ Assume that the basis  $B$  is optimal:

$$\begin{aligned} \text{maximize } z &= \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{b} + [\mathbf{c}_N^T - \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{N}] \mathbf{x}_N \\ \text{subject to } &\mathbf{B}^{-1} \mathbf{b} - \mathbf{B}^{-1} \mathbf{N} \mathbf{x}_N \geq \mathbf{0}^m, \\ &\mathbf{x}_N \geq \mathbf{0}^{n-m} \end{aligned}$$

- ▶  $\mathbf{x}_B = \mathbf{B}^{-1} \mathbf{b} - \mathbf{B}^{-1} \mathbf{N} \mathbf{x}_N$



## Changes in the right hand side coefficients

- ▶ Suppose  $\mathbf{b}$  changes to  $\mathbf{b} + \Delta\mathbf{b}$

⇒ New optimal value:

$$z^{\text{new}} = \mathbf{c}_B^T \mathbf{B}^{-1} (\mathbf{b} + \Delta\mathbf{b}) = z + \mathbf{c}_B^T \mathbf{B}^{-1} \Delta\mathbf{b}$$

- ▶ The current basis is feasible if  $\mathbf{B}^{-1}(\mathbf{b} + \Delta\mathbf{b}) \geq 0$
- ▶ If not: negative values will occur in the right hand side
- ▶ The reduced costs are unchanged (negative, at optimum)  
⇒ this can be resolved using the *dual simplex method*



## Changes in the right hand side coefficients

- ▶ Consider the linear program

$$\begin{array}{ll} \text{minimize } z = & -x_1 \quad -2x_2 \\ \text{subject to} & -2x_1 \quad +x_2 \leq 2 \\ & -x_1 \quad +2x_2 \leq 7 \\ & x_1 \leq 3 \\ & x_1, x_2 \geq 0 \end{array}$$

DRAW GRAPH!!

- ▶ The optimal solution is given by

basis	-z	x <sub>1</sub>	x <sub>2</sub>	s <sub>1</sub>	s <sub>2</sub>	s <sub>3</sub>	RHS
-z	1	0	0	0	1	2	13
x <sub>2</sub>	0	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	5
x <sub>1</sub>	0	1	0	0	0	1	3
s <sub>1</sub>	0	0	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	3



## Changes in the right hand side coefficients

- ▶ Change the right hand side according to

$$\begin{array}{ll} \text{minimize } z = & -x_1 \quad -2x_2 \\ \text{subject to} & -2x_1 \quad +x_2 \leq 2 \\ & -x_1 \quad +2x_2 \leq 7 + \delta \\ & x_1 \leq 3 \\ & x_1, x_2 \geq 0 \end{array}$$

- ▶ The change in the right hand side is given by  
 $\mathbf{B}^{-1}(0, \delta, 0)^T = (\frac{1}{2}\delta, 0, -\frac{1}{2}\delta)^T \Rightarrow$  new optimal tableau:

basis	-z	x <sub>1</sub>	x <sub>2</sub>	s <sub>1</sub>	s <sub>2</sub>	s <sub>3</sub>	RHS
-z	1	0	0	0	1	2	13 + $\delta$
x <sub>2</sub>	0	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	5 + $\frac{1}{2}\delta$
x <sub>1</sub>	0	1	0	0	0	1	3
s <sub>1</sub>	0	0	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	3 - $\frac{1}{2}\delta$

- ▶ The current basis is feasible if  $-10 \leq \delta \leq 6$



## Changes in the right hand side coefficients

- ▶ Suppose  $\delta = 8$ :

basis	-z	x <sub>1</sub>	x <sub>2</sub>	s <sub>1</sub>	s <sub>2</sub>	s <sub>3</sub>	RHS
-z	1	0	0	0	1	2	21
x <sub>2</sub>	0	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	9
x <sub>1</sub>	0	1	0	0	0	1	3
s <sub>1</sub>	0	0	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	-1

- ▶ Dual simplex iteration:
- ▶  $s_1 = -1$  has to leave the basis
- ▶ Find the smallest ratio between reduced costs (for non-basic columns) and (negative) elements in the "s<sub>1</sub>-row" (to stay optimal)
- ▶  $s_2$  will enter the basis — **New optimal** tableau:

basis	-z	x <sub>1</sub>	x <sub>2</sub>	s <sub>1</sub>	s <sub>2</sub>	s <sub>3</sub>	RHS
-z	1	0	0	2	0	5	19
x <sub>2</sub>	0	0	1	1	0	2	8
x <sub>1</sub>	0	1	0	0	0	1	3
s <sub>2</sub>	0	0	0	-2	1	-3	2



## Changes in the objective coefficients

- ▶ Suppose  $\mathbf{c}$  changes to  $\mathbf{c} + \Delta\mathbf{c}$

- ▶ The new optimal value:

$$z^{\text{new}} = (\mathbf{c}_B + \Delta\mathbf{c}_B)^T \mathbf{B}^{-1} \mathbf{b} = z + \Delta\mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{b}$$

- ▶ The current basis is optimal if  $(\mathbf{c}_N + \Delta\mathbf{c}_N)^T - (\mathbf{c}_B + \Delta\mathbf{c}_B)^T \mathbf{B}^{-1} \mathbf{N} \leq \mathbf{0}$

- ▶ If not: more simplex iterations to find the optimal solution

## Changes in the objective coefficients

- ▶ Suppose  $\delta = 4$ : new tableau:

basis	-z	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	RHS
-z	1	0	0	0	-1	0	-7
$x_2$	0	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	5
$x_1$	0	1	0	0	0	1	3
$s_1$	0	0	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	3

- ▶ Let  $s_2$  enter and  $x_2$  leave the basis. New optimal tableau:

basis	-z	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	RHS
-z	1	0	2	0	0	1	3
$s_2$	0	0	2	0	1	1	10
$x_1$	0	1	0	0	0	1	3
$s_1$	0	0	1	1	0	2	8

## Changes in the objective coefficients

- ▶ Change the objective according to

$$\begin{aligned} \text{minimize } z &= -x_1 + (-2 + \delta)x_2 \\ \text{subject to } & -2x_1 + x_2 \leq 2 \\ & -x_1 + 2x_2 \leq 7 \\ & x_1 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

- ▶ The changes in the reduced costs are given by  $-(\delta, 0, 0)\mathbf{B}^{-1}\mathbf{N} = (-\frac{1}{2}\delta, -\frac{1}{2}\delta) \Rightarrow$  new optimal tableau:

basis	-z	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	RHS
-z	1	0	0	0	$1 - \frac{1}{2}\delta$	$2 - \frac{1}{2}\delta$	$13 - 5\delta$
$x_2$	0	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	5
$x_1$	0	1	0	0	0	1	3
$s_1$	0	0	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	3

- ▶ The current basis is optimal if  $\delta \leq 2$