MVE165/MMG630, Applied Optimization
Lecture 6
Minimum cost flow models and algorithms

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Linear programming formulation of MG Auto

- Variables: x_{ij} = number of cars sent from plant i to distribution center j

$$\min z = 80x_{11}+215x_{12}+100x_{21}+108x_{22}+102x_{31}+68x_{32}$$

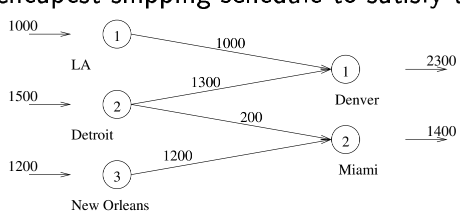
$$\begin{array}{lcccccccll}
\text{s.t.} & x_{11} & +x_{12} & & & & & = & 1000 & \text{(LA)} \\
 & & & x_{21} & +x_{22} & & & = & 1500 & \text{(Detr)} \\
 & & & & & x_{31} & +x_{32} & = & 1200 & \text{(NO)} \\
 & x_{11} & & +x_{21} & & +x_{31} & & = & 2300 & \text{(Den)} \\
 & & x_{12} & & +x_{22} & & +x_{32} & = & 1400 & \text{(Mi)} \\
 & x_{11}, & x_{12}, & x_{21}, & x_{22}, & x_{31}, & x_{32} & \geq & 0 &
\end{array}$$

Transportation models: An example

- MG Auto has three plants, LA, Detroit, New Orleans, and two distribution centers, Denver and Miami
- Capacities of the plants: 1000, 1500, and 1200 cars
- Demands at distributions centers: 2300 and 1400 cars
- Transportation cost per car between plants and centers:

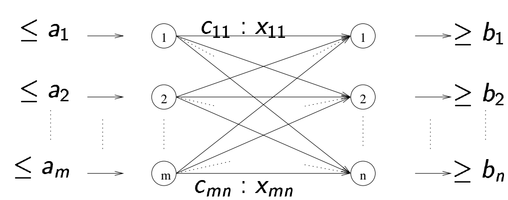
	Denver	Miami
LA	\$80	\$215
Detroit	\$100	\$108
New Orleans	\$102	\$68

- Find the cheapest shipping schedule to satisfy the demand



Definition of the transportation model

- m sources and n destinations $\Leftrightarrow$ **nodes**
- a_i = amount of supply at source (node) i
- b_j = amount of demand at destination (node) j
- **Arc** (i,j) $\Leftrightarrow$ connection from source i to destination j
- c_{ij} = cost per unit of flow on arc (i,j)
- **Variables:** x_{ij} = amount of goods shipped on arc (i,j)
- **Objective:** find $x_{ij} \geq 0$ such that the total cost is minimized while satisfying all supply and demand restrictions



Linear programming transportation model

$$
\begin{array}{llcll}
\min z := & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} & & & \\
\text{s.t.} & \sum_{j=1}^{n} x_{ij} & \leq & a_i, \quad i=1,\dots,m & \text{(supply)} \\
 & \sum_{i=1}^{m} x_{ij} & \geq & b_j, \quad j=1,\dots,n & \text{(demand)} \\
 & x_{ij} & \geq & 0, \quad i=1,\dots,m, \quad j=1,\dots,n &
\end{array}
$$

- Feasible solutions exist *if and only if* $\boxed{\sum_i a_i \geq \sum_j b_j}$
- The constraint matrix has special properties (totally unimodular) ⇒ integer solutions in extreme points of the feasible polyhedron
- This property holds for all problems that can be formulated as linear flows in networks

Optimal solution of the MG Auto model

	Denver	Miami	Supply
LA	\$80 / 1000	\$215 / 0	1000
Detroit	\$100 / 1300	\$108 / 200	1500
New Orleans	\$102 / 0	\$68 / 1200	1200
Demand	2300	1400	

- Solution method: A special version of the simplex method:

⇒ Equality constraints are required

Further details of the transportation model

- The transportation model is a **special case** of linear programming.
- Representation in a **transportation tableau**:

	Denver	Miami	Supply
LA	\$80 / x_{11}	\$215 / x_{12}	1000
Detroit	\$100 / x_{21}	\$108 / x_{22}	1500
New Orleans	\$102 / x_{31}	\$68 / x_{32}	1200
Demand	2300	1400	

Further details of the transportation model

- What if total amount of demand $\neq$ total amount of supply? ($\sum_i a_i > \sum_j b_j$ (feasible) or $\sum_i a_i < \sum_j b_j$ (infeasible))

$$
\boxed{
\begin{array}{llcll}
\min z := & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} & & & \\
\text{s.t.} & \sum_{j=1}^{n} x_{ij} & \leq & a_i, & i=1,\dots,m \\
 & \sum_{i=1}^{m} x_{ij} & \geq & b_j, & j=1,\dots,n \\
 & x_{ij} & \geq & 0, & i=1,\dots,m, j=1,\dots,n
\end{array}}
$$

⇒ **Balance** the model by dummy source $(m+1)$ or destination $(n+1)$

- Suppose $\sum_i a_i > \sum_j b_j \Rightarrow$ Let $b_{n+1} := \sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j$

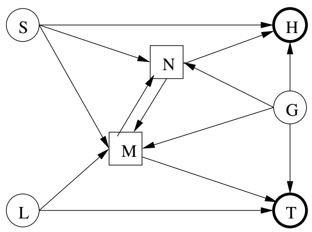
⇒ **Balanced** transportation model—equality constraints

$$
\boxed{
\begin{array}{llcll}
\min z := & \sum_{i=1}^{m}\sum_{j=1}^{n+1} c_{ij}x_{ij} & & & \\
\text{s.t.} & \sum_{j=1}^{n+1} x_{ij} & = & a_i, & i=1,\dots,m \\
 & \sum_{i=1}^{m} x_{ij} & = & b_j, & j=1,\dots,n+1 \\
 & x_{ij} & \geq & 0, & i=1,\dots,m, j=1,\dots,n+1
\end{array}}
$$

Minimum cost flow in a general network: An example

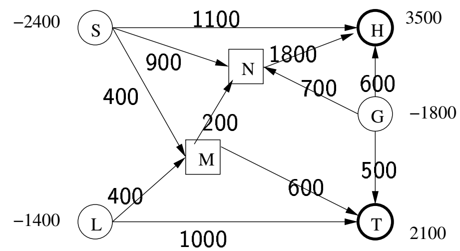
- Two paper mills: Holmsund and Tuna
- Three saw mills: Silje, Graninge and Lunden
- Two storage terminals: Norrstig and Mellansel

Facility	Supply (m^3)	Demand (m^3)
Silje	2400	
Graninge	1800	
Lunden	1400	
Holmsund		3500
Tuna		2100



Minimum cost flow in a general network: An example

- Objective: Minimize transportation costs
- Satisfy demand
- Do not exceed the supply
- Do not exceed the transportation capacities
- An optimal solution



Minimum cost flow in a general network: An example

- Transportation opportunities:

From	To	Price/m^3	Capacity (m^3)
Silje	Norrstig	20	900
Silje	Mellansel	26	1000
Silje	Holmsund	45	1100
Graninge	Norrstig	8	700
Graninge	Mellansel	14	900
Graninge	Holmsund	37	600
Graninge	Tuna	22	600
Lunden	Mellansel	32	600
Lunden	Tuna	23	1000
Norrstig	Holmsund	11	1800
Norrstig	Mellansel	9	1800
Mellansel	Norrstig	9	1800
Mellansel	Tuna	9	1800

Minimum cost flow in a general network: An example

$$
\begin{array}{llcrl}
\min z := & \multicolumn{4}{l}{20x_{SN} + 26x_{SM} + 45x_{SH} + 8x_{GN} + 14x_{GM}} \\
& \multicolumn{4}{l}{+37x_{GH} + 22x_{GT} + 32x_{LM} + 23x_{LT} + 11x_{NH}} \\
& \multicolumn{4}{l}{+9x_{NM} + 9x_{MN} + 9x_{MT}} \\
\text{subject to} & -x_{SN} - x_{SM} - x_{SH} & = & -2400 & \text{(Silje)} \\
& -x_{GN} - x_{GM} - x_{GH} - x_{GT} & = & -1800 & \text{(Graninge)} \\
& -x_{LM} - x_{LT} & = & -1400 & \text{(Lunden)} \\
& x_{SN} + x_{GN} + x_{MN} - x_{NM} - x_{NH} & = & 0 & \text{(Norrstig)} \\
& x_{SM} + x_{LM} + x_{GM} + x_{NM} - x_{MN} - x_{MT} & = & 0 & \text{(Mellansel)} \\
& x_{SH} + x_{GH} + x_{NH} & = & 3500 & \text{(Holmsund)} \\
& x_{GT} + x_{LT} + x_{MT} & = & 2100 & \text{(Tuna)} \\
& 0 \le x_{SN} & \le & 900 & \\
& 0 \le x_{SM} & \le & 1000 & \\
& 0 \le x_{SH} & \le & 1100 & \\
& 0 \le x_{GN} & \le & 700 & \\
& 0 \le x_{GM} & \le & 900 & \\
& 0 \le x_{GH} & \le & 600 & \\
& 0 \le x_{GT} & \le & 600 & \\
& 0 \le x_{LM} & \le & 600 & \\
& 0 \le x_{LT} & \le & 1000 & \\
& 0 \le x_{NH} & \le & 1800 & \\
& 0 \le x_{NM} & \le & 1800 & \\
& 0 \le x_{MN} & \le & 1800 & \\
& 0 \le x_{MT} & \le & 1800 &
\end{array}
$$

- The columns $\mathbf{A}_j$ of the equality constraint matrix ($\mathbf{Ax} = \mathbf{b}$) have one 1-element, one -1-element; the remaining elements are 0

The transportation problem: primal and dual problems

[Primal]

$$
\begin{aligned}
\min z := \quad & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \\
\text{subject to} \quad & \sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1,\dots,m \\
& \sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1,\dots,n \\
& x_{ij} \geq 0, \quad i = 1,\dots,m, j = 1,\dots,n
\end{aligned}
$$

[Dual]

$$
\begin{aligned}
\max w := \quad & \sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j \\
\text{subject to} \quad & u_i + v_j \leq c_{ij}, \quad i = 1,\dots,m, j = 1,\dots,n
\end{aligned}
$$

Step 1: Finding a feasible solution

- The Northwest-corner method

					Supply
	10 **5** →	2 **10**	20	11	15
	12	↓ 7 **5** →	9 **15** →	20 **5**	25
	4	14	16	↓ 18 **10**	10
Demand	5	15	15	15	

- Cost for this solution:
  $10 \cdot 5 + 2 \cdot 10 + 7 \cdot 5 + 9 \cdot 15 + 20 \cdot 5 + 18 \cdot 10 = 520$
- The nonzero variables are *basic* variables
- A *basis* for a network flow problem forms a *tree* in the corresponding graph
- DRAW THIS GRAPH!

The Simplex algorithm for transportation problems (generalized for general minimum cost flows in Ch. 10.7)

- The algorithm follows the steps of the Simplex method
- Transportation tableau instead of simplex tableau
- The special structure allows for simpler operations
- As with the Simplex method: First find a feasible solution
- Iteratively improve this solution with pivot operations until an optimal solution is found

  The transportation algorithm:

  1. Find a feasible solution
  2. Find the *entering variable*: use simplex *optimality condition*
     If optimality condition holds: stop. Else go to step 3
  3. Find the *leaving variable*: use *simplex feasibility condition*
     Go to step 2

Step 2: Finding an entering variable

- Reduced cost computations: $\bar{\mathbf{c}}$
  (Recall, lecture 3: $\bar{c}_j = c_j - \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{A}_j$)
- Here: $\bar{c}_{ij} = c_{ij} - u_i - v_j$
- Basic variables: $\bar{c}_{ij} = 0 \Rightarrow$ values for u_i and v_j
- Non-basic variables: $\bar{c}_{ij} < 0 \Rightarrow x_{ij}$ candidate for entering the basis

multipliers	$v_1 = 10$	$v_2 = 2$	$v_3 = 4$	$v_4 = 15$	Supply
$u_1 = 0$	10 **5** →	2 **10**	20	11	15
$u_2 = 5$	12	↓ 7 **5** →	9 **15** →	20 **5**	25
$u_3 = 3$	4	14	16	↓ 18 **10**	10
Demand	5	15	15	15	

Step 2: Finding an entering variable

- Here: $\overline{c}_{ij} = c_{ij} - u_i - v_j$ $\quad \boxed{\overline{c}_{ij}}$

multipliers	$v_1 = 10$	$v_2 = 2$	$v_3 = 4$	$v_4 = 15$	Supply
$u_1 = 0$	10 / **5**	2 / **10**	20 / $\boxed{16}$	11 / $\boxed{-4}$	15
$u_2 = 5$	12 / $\boxed{-3}$	7 / **5**	9 / **15**	20 / **5**	25
$u_3 = 3$	4 / $\boxed{\mathbf{-9}}$	14 / $\boxed{9}$	16 / $\boxed{9}$	18 / **10**	10
Demand	5	15	15	15	

- Entering variable: x_{31}

Update variable values

- A degenerate solution: basic variable $x_{22} = 0$
- Update also dual solution u_i & v_j such that $\overline{c}_{ij} = 0$ for basic variables ($\overline{c}_{31} = c_{31} - u_3 - v_1 = 0$ but $\overline{c}_{11} = c_{11} - u_1 - v_1 = 9 > 0$)

multipliers	$v_1 = 1$	$v_2 = 2$	$v_3 = 4$	$v_4 = 15$	Supply
$u_1 = 0$	10 / $\boxed{9}$	2 / **15**	20 / $\boxed{16}$	11 / $\boxed{\mathbf{-4}}$	15
$u_2 = 5$	12 / $\boxed{6}$	7 / **0**	9 / **15**	20 / **10**	25
$u_3 = 3$	4 / **5**	14 / $\boxed{9}$	16 / $\boxed{9}$	18 / **5**	10
Demand	5	15	15	15	

- Entering variable: x_{14}
- Continue on the board ...

Step 3: Finding the leaving variable

- One of the basic variables has to leave the basis
- Select the variable according the the simplex feasibility condition ($x_{ij} \geq 0$ for all i, j)
- $x_{31} := \Theta \Rightarrow \ldots$ A cycle in the graph: DRAW!

multipliers	$v_1 = 10$	$v_2 = 2$	$v_3 = 4$	$v_4 = 15$	Supply
$u_1 = 0$	10 / **5**–Θ	2 / **10**+Θ	20 / $\boxed{16}$	11 / $\boxed{-4}$	15
$u_2 = 5$	12 / $\boxed{-3}$	7 / **5**–Θ	9 / **15**	20 / **5**+Θ	25
$u_3 = 3$	4 / +Θ $\boxed{\mathbf{-9}}$	14 / $\boxed{9}$	16 / $\boxed{9}$	18 / **10**–Θ	10
Demand	5	15	15	15	

- $x_{ij} \geq 0 \Rightarrow \Theta \leq 5 \Rightarrow$ Choose $\Theta = 5$

The assignment model

- A special case of the transportation model
- Given n persons and n jobs
- Given further the cost c_{ij} of assigning person i to job j
- Binary variables $x_{ij} = 1$ if person i does job j and $x_{ij} = 0$ otherwise
- Find the cheapest assignment of persons to jobs such that all jobs are done

$$\begin{array}{lrcll} \min & \sum_{ij} c_{ij} x_{ij} & & & \\ \text{s.t.} & \sum_j x_{ij} & = & 1 & \forall i \\ & \sum_i x_{ij} & = & 1 & \forall j \\ & x_{ij} & \geq & 0 & \forall i, j \end{array}$$

- The optimal solution is binary (due to the totally unimodular constraint matrix)

An assignment example

- 3 children: John, Karen and Terri
- 3 tasks: mow, paint and wash.
- Given further a "cost" (time, uncomfort,...) for each combination of child/task
- How should the parents distribute the tasks to minimize the cost?

	Mow	Paint	Wash
John	15	10	9
Karen	9	15	10
Terri	10	12	8

- Choose exactly one element in each row and one in each column
- What if there are more tasks than children or vice versa?