MVE165/MMG630, Applied Optimization
Lecture 6
## Minimum cost flow models and algorithms

Ann-Brith Strömberg

2009–03–27

Transportation models: An example

- MG Auto has three plants, LA, Detroit, New Orleans, and two distribution centers, Denver and Miami
- Capacities of the plants: 1000, 1500, and 1200 cars
- Demands at distributions centers: 2300 and 1400 cars
- Transportation cost per car between plants and centers:

	Denver	Miami
LA	$80	$215
Detroit	$100	$108
New Orleans	$102	$68

- Find the cheapest shipping schedule to satisfy the demand
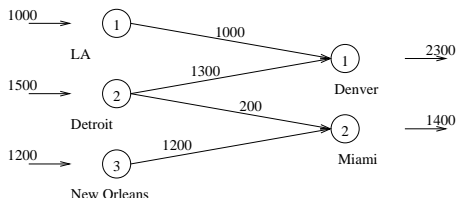
Linear programming formulation of MG Auto

- Variables: x_{ij} = number of cars sent from plant i to distribution center j

min z= $80x_{11}$+$215x_{12}$+$100x_{21}$+$108x_{22}$+$102x_{31}$+$68x_{32}$

s.t.	x_{11}	$+x_{12}$					$=$ 1000	(LA)
			x_{21}	$+x_{22}$			$=$ 1500	(Detr)
					x_{31}	$+x_{32}$	$=$ 1200	(NO)
	x_{11}		$+x_{21}$		$+x_{31}$		$=$ 2300	(Den)
		x_{12}		$+x_{22}$		$+x_{32}$	$=$ 1400	(Mi)
	$x_{11},$	$x_{12},$	$x_{21},$	$x_{22},$	$x_{31},$	x_{32} $\geq$	0	

Definition of the transportation model

- m sources and n destinations $\Leftrightarrow$ **nodes**
- $a_i =$ amount of supply at source (node) i
- $b_j =$ amount of demand at destination (node) j
- **Arc** $(i, j) \Leftrightarrow$ connection from source i to destination j
- $c_{ij} =$ cost per unit of flow on arc (i, j)
- **Variables:** $x_{ij} =$ amount of goods shipped on arc (i, j)
- **Objective:** find $x_{ij} \geq 0$ such that the total cost is minimized while satisfying all supply and demand restrictions
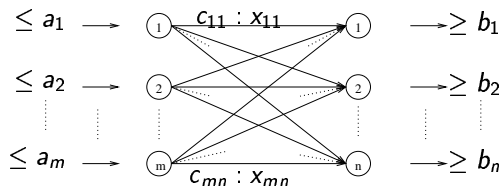
Linear programming transportation model

$$\min z := \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} x_{ij} \leq a_i, \quad i = 1, \ldots, m \quad \text{(supply)}$$

$$\sum_{i=1}^{m} x_{ij} \geq b_j, \quad j = 1, \ldots, n \quad \text{(demand)}$$

$$x_{ij} \geq 0, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n$$

▶ Feasible solutions exist *if and only if* $\boxed{\sum_i a_i \geq \sum_j b_j}$

▶ The constraint matrix has special properties (totally unimodular) $\Rightarrow$ integer solutions in extreme points of the feasible polyhedron

▶ This property holds for all problems that can be formulated as linear flows in networks

Further details of the transportation model

- The transportation model is a **special case** of linear programming.
- Representation in a **transportation tableau**:

	Denver	Miami	Supply
LA	$80 x_{11}	$215 x_{12}	1000
Detroit	$100 x_{21}	$108 x_{22}	1500
New Orleans	$102 x_{31}	$68 x_{32}	1200
Demand	2300	1400	

Optimal solution of the MG Auto model

	Denver	Miami	Supply
LA	$80	$215	
	1000	0	1000
Detroit	$100	$108	
	1300	200	1500
New Orleans	$102	$68	
	0	1200	1200
Demand	2300	1400	

▶ Solution method: A special version of the simplex method:

⇒ Equality constraints are required

Further details of the transportation model

▶ What if total amount of demand $\neq$ total amount of supply?
($\sum_i a_i > \sum_j b_j$ (feasible) or $\sum_i a_i < \sum_j b_j$ (infeasible))

$$
\begin{array}{lrcll}
\min z := & \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij} & & \\
\text{s.t.} & \sum_{j=1}^n x_{ij} & \leq & a_i, & i = 1, \dots, m \\
& \sum_{i=1}^m x_{ij} & \geq & b_j, & j = 1, \dots, n \\
& x_{ij} & \geq & 0, & i = 1, \dots, m, j = 1, \dots, n
\end{array}
$$

$\Rightarrow$ **Balance** the model by dummy source $(m + 1)$ or destination $(n + 1)$

▶ Suppose $\sum_i a_i > \sum_j b_j \Rightarrow$ Let $b_{n+1} := \sum_{i=1}^m a_i - \sum_{j=1}^n b_j$

$\Rightarrow$ **Balanced** transportation model—equality constraints

$$
\begin{array}{lrcll}
\min z := & \sum_{i=1}^m \sum_{j=1}^{n+1} c_{ij} x_{ij} & & \\
\text{s.t.} & \sum_{j=1}^{n+1} x_{ij} & = & a_i, & i = 1, \dots, m \\
& \sum_{i=1}^m x_{ij} & = & b_j, & j = 1, \dots, n + 1 \\
& x_{ij} & \geq & 0, & i{=}1, \dots, m, j{=}1, \dots, n{+}1
\end{array}
$$

Minimum cost flow in a general network: An example

- Two paper mills: Holmsund and Tuna
- Three saw mills: Silje, Graninge and Lunden
- Two storage terminals: Norrstig and Mellansel

Facility	Supply (m^3)	Demand (m^3)
Silje	2400	
Graninge	1800	
Lunden	1400	
Holmsund		3500
Tuna		2100
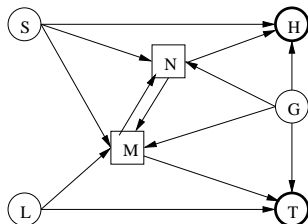
Minimum cost flow in a general network: An example

- Transportation opportunities:

From	To	Price/m^3	Capacity (m^3)
Silje	Norrstig	20	900
Silje	Mellansel	26	1000
Silje	Holmsund	45	1100
Graninge	Norrstig	8	700
Graninge	Mellansel	14	900
Graninge	Holmsund	37	600
Graninge	Tuna	22	600
Lunden	Mellansel	32	600
Lunden	Tuna	23	1000
Norrstig	Holmsund	11	1800
Norrstig	Mellansel	9	1800
Mellansel	Norrstig	9	1800
Mellansel	Tuna	9	1800

▶ Objective: Minimize transportation costs

▶ ~~Satisfy demand~~

▶ Do not exceed the supply

▶ Do not exceed the transportation capacities

▶ An optimal solution
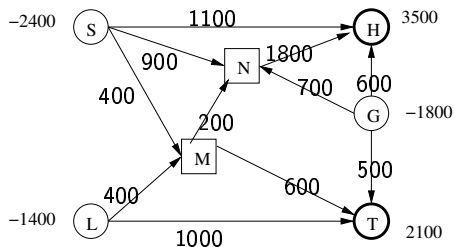
$$\min z := \quad 20x_{SN} + 26x_{SM} + 45x_{SH} + 8x_{GN} + 14x_{GM}$$
$$+37x_{GH} + 22x_{GT} + 32x_{LM} + 23x_{LT} + 11x_{NH}$$
$$+9x_{NM} + 9x_{MN} + 9x_{MT}$$

subject to

$$
\begin{array}{rcll}
-x_{SN} - x_{SM} - x_{SH} & = & -2400 & \text{(Silje)} \\
-x_{GN} - x_{GM} - x_{GH} - x_{GT} & = & -1800 & \text{(Graninge)} \\
-x_{LM} - x_{LT} & = & -1400 & \text{(Lunden)} \\
x_{SN} + x_{GN} + x_{MN} - x_{NM} - x_{NH} & = & 0 & \text{(Norrstig)} \\
x_{SM} + x_{LM} + x_{GM} + x_{NM} - x_{MN} - x_{MT} & = & 0 & \text{(Mellansel)} \\
x_{SH} + x_{GH} + x_{NH} & = & 3500 & \text{(Holmsund)} \\
x_{GT} + x_{LT} + x_{MT} & = & 2100 & \text{(Tuna)}
\end{array}
$$

$$
\begin{array}{ccccr}
0 & \leq & x_{SN} & \leq & 900 \\
0 & \leq & x_{SM} & \leq & 1000 \\
0 & \leq & x_{SH} & \leq & 1100 \\
0 & \leq & x_{GN} & \leq & 700 \\
0 & \leq & x_{GM} & \leq & 900 \\
0 & \leq & x_{GH} & \leq & 600 \\
0 & \leq & x_{GT} & \leq & 600 \\
0 & \leq & x_{LM} & \leq & 600 \\
0 & \leq & x_{LT} & \leq & 1000 \\
0 & \leq & x_{NH} & \leq & 1800 \\
0 & \leq & x_{NM} & \leq & 1800 \\
0 & \leq & x_{MN} & \leq & 1800 \\
0 & \leq & x_{MT} & \leq & 1800
\end{array}
$$

▶ The columns $\mathbf{A}_j$ of the equality constraint matrix ($\mathbf{Ax} = \mathbf{b}$) have one 1-element, one -1-element; the remaining elements are 0

[Primal]

$$\min z := \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\text{subject to} \quad \sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, \ldots, m$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, \ldots, n$$

$$x_{ij} \geq 0, \quad i = 1, \ldots, m, j = 1, \ldots, n$$

[Dual]

$$\max w := \sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j$$

$$\text{subject to} \quad u_i + v_j \leq c_{ij}, \quad i = 1, \ldots, m, j = 1, \ldots, n$$

The Simplex algorithm for transportation problems (generalized for general minimum cost flows in Ch. 10.7)

- The algorithm follows the steps of the Simplex method
- Transportation tableau instead of simplex tableau
- The special structure allows for simpler operations
- As with the Simplex method: First find a feasible solution
- Iteratively improve this solution with pivot operations until an optimal solution is found

The transportation algorithm:

1. Find a feasible solution
2. Find the *entering variable*: use simplex *optimality condition*
   If optimality condition holds: stop. Else go to step 3
3. Find the *leaving variable*: use simplex *feasibility condition*
   Go to step 2

- The Northwest-corner method

Supply

	10		2		20		11	
5	$\rightarrow$	**10**						15
	12	$\downarrow$	7		9		20	
		5	$\rightarrow$	**15**	$\rightarrow$	**5**		25
	4		14		16	$\downarrow$	18	
						10		10

Demand   5   15   15   15

- Cost for this solution:
  $10 \cdot 5 + 2 \cdot 10 + 7 \cdot 5 + 9 \cdot 15 + 20 \cdot 5 + 18 \cdot 10 = 520$
- The nonzero variables are *basic* variables
- A *basis* for a network flow problem forms a *tree* in the corresponding graph
- DRAW THIS GRAPH!

Step 2: Finding an entering variable

- Reduced cost computations: $\overline{\mathbf{c}}$
  (Recall, lecture 3: $\overline{c}_j = c_j - \mathbf{c}_B^{\mathrm{T}} \mathbf{B}^{-1} \mathbf{A}_j$)
- Here: $\overline{c}_{ij} = c_{ij} - u_i - v_j$
- Basic variables: $\overline{c}_{ij} = 0 \Rightarrow$ values for u_i and v_j
- Non-basic variables: $\overline{c}_{ij} < 0 \Rightarrow x_{ij}$ candidate for entering the basis

multipliers	$v_1 = 10$	$v_2 = 2$	$v_3 = 4$	$v_4 = 15$	Supply
	10	2	20	11	
$u_1 = 0$	**5** $\rightarrow$	**10**			15
	12	$\downarrow$ 7	9	20	
$u_2 = 5$		**5** $\rightarrow$	**15** $\rightarrow$	**5**	25
	4	14	16	$\downarrow$ 18	
$u_3 = 3$				**10**	10
Demand	5	15	15	15	

- Here: $\overline{c}_{ij} = c_{ij} - u_i - v_j$ $\boxed{\overline{c}_{ij}}$

multipliers	$v_1 = 10$	$v_2 = 2$	$v_3 = 4$	$v_4 = 15$	Supply
$u_1 = 0$	10 **5**	2 **10**	20 $\boxed{16}$	11 $\boxed{-4}$	15
$u_2 = 5$	12 $\boxed{-3}$	7 **5**	9 **15**	20 **5**	25
$u_3 = 3$	4 $\boxed{\mathbf{-9}}$	14 $\boxed{9}$	16 $\boxed{9}$	18 **10**	10
Demand	5	15	15	15	

- Entering variable: x_{31}

▶ One of the basic variables has to leave the basis

▶ Select the variable according the the simplex feasibility condition ($x_{ij} \geq 0$ for all i, j)

▶ $x_{31} := \Theta \Rightarrow \ldots$ A cycle in the graph: DRAW!

multipliers	$v_1 = 10$	$v_2 = 2$	$v_3 = 4$	$v_4 = 15$	Supply
	10	2	20	11	
$u_1 = 0$	**5**−Θ	**10**+Θ	**16** (boxed)	**−4** (boxed)	15
	12	7	9	20	
$u_2 = 5$	**−3** (boxed)	**5**−Θ	**15**	**5**+Θ	25
	4	14	16	18	
$u_3 = 3$	+Θ **−9** (boxed)	**9** (boxed)	**9** (boxed)	**10**−Θ	10
Demand	5	15	15	15	

▶ $x_{ij} \geq 0 \Rightarrow \Theta \leq 5 \Rightarrow$ Choose $\Theta = 5$

Update variable values

- A degenerate solution: basic variable $x_{22} = 0$
- Update also dual solution u_i & v_j such that $\overline{c}_{ij} = 0$ for basic variables ($\overline{c}_{31} = c_{31} - u_3 - v_1 = 0$ but $\overline{c}_{11} = c_{11} - u_1 - v_1 = 9 > 0$)

multipliers	$v_1 = 1$	$v_2 = 2$	$v_3 = 4$	$v_4 = 15$	Supply
	10	2	20	11	
$u_1 = 0$	$\boxed{9}$	**15**	$\boxed{16}$	$\boxed{-4}$	15
	12	7	9	20	
$u_2 = 5$	$\boxed{6}$	**0**	**15**	**10**	25
	4	14	16	18	
$u_3 = 3$	**5**	$\boxed{9}$	$\boxed{9}$	**5**	10
Demand	5	15	15	15	

- Entering variable: x_{14}
- Continue on the board ...

- A special case of the transportation model
- Given n persons and n jobs
- Given further the cost c_{ij} of assigning person i to job j
- Binary variables $x_{ij} = 1$ if person i does job j and $x_{ij} = 0$ otherwise
- Find the cheapest assignment of persons to jobs such that all jobs are done

$$
\begin{aligned}
\min \quad & \sum_{ij} c_{ij} x_{ij} \\
\text{s.t.} \quad & \sum_j x_{ij} = 1 \quad \forall i \\
& \sum_i x_{ij} = 1 \quad \forall j \\
& x_{ij} \geq 0 \quad \forall i, j
\end{aligned}
$$

- The optimal solution is binary (due to the totally unimodular constraint matrix)

- 3 children: John, Karen and Terri
- 3 tasks: mow, paint and wash.
- Given further a "cost" (time, uncomfort,...) for each combination of child/task
- How should the parents distribute the tasks to minimize the cost?

	Mow	Paint	Wash
John	15	10	9
Karen	9	15	10
Terri	10	12	8

- Choose exactly one element in each row and one in each column
- What if there are more tasks than children or vice versa?