MVE165/MMG630, Applied Optimization
Lecture 8
Integer linear programming algorithms

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2009–03–31

Relaxations and feasible solutions

- Consider a maximization integer linear program (ILP):

$$[\text{ILP}] \qquad z^* = \quad \begin{array}{rrcll} \max & \mathbf{c}^{\mathrm{T}}\mathbf{x} & & & \\ \text{subject to} & \mathbf{A}\mathbf{x} & \leq & \mathbf{b} & \\ & \mathbf{x} & \geq & \mathbf{0} & \text{and integer} \end{array}$$

- The feasible set $X = \{\mathbf{x} \in Z_+^n \,|\, \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$
- How prove that a solution $\mathbf{x}^* \in X$ is optimal?
- We cannot use strong duality/complementarity as for linear programming (where X is convex (polyhedral))!
- Bounds on the optimal value
  - Optimistic estimate $\bar{z} \geq z^*$ from a *relaxation* of ILP
  - Pessimistic estimate $\underline{z} \leq z^*$ from a *feasible solution* to ILP
- Goal: Find tight bounds for z^*: $\bar{z} - \underline{z} \leq \varepsilon$ and $\varepsilon > 0$ "small"

Optimistic estimates of z^* from relaxations

- Either: Enlarge the set X by removing constraints
- Or: Replace $\mathbf{c}^{\mathrm{T}}\mathbf{x}$ by an overestimating function f, i.e., such that $f(\mathbf{x}) \geq \mathbf{c}^{\mathrm{T}}\mathbf{x}$ for all $\mathbf{x} \in X$
- Or: Do both

$\Rightarrow$ solve a *relaxation* of (ILP)

- Example (enlarge X):

  $X = \{\mathbf{x} \geq \mathbf{0} \mid \mathbf{A}\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \text{ integer } \}$ and
  $X^{\mathrm{LP}} = \{\mathbf{x} \geq \mathbf{0} \mid \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$

$$\Rightarrow \quad z^{\mathrm{LP}} = \max_{\mathbf{x} \in X^{\mathrm{LP}}} \mathbf{c}^{\mathrm{T}}\mathbf{x}$$

- It holds that $z^{\mathrm{LP}} \geq z^*$ since $X \subseteq X^{\mathrm{LP}}$

Relaxation principles that yield more tractable problems

- *Linear programming relaxation*
  Remove integrality requirements (enlarge X)

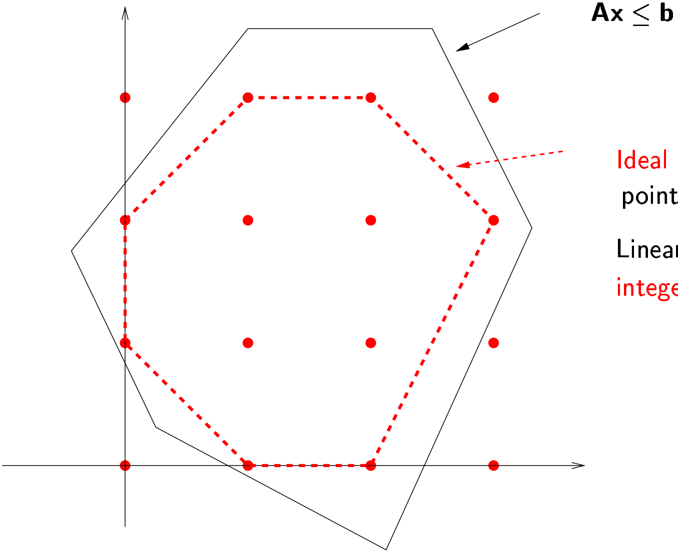
- *Combinatorial relaxation*
  E.g. remove subcycle constraints from asymmetric TSP $\Rightarrow$ min-cost assignment (enlarge X)

- *Lagrangean relaxation*
  Move "complicating" constraints to the objective function, with penalties for infeasible solutions; then find "optimal" penalties (enlarge X and find $f(\mathbf{x}) \geq \mathbf{c}^{\mathrm{T}}\mathbf{x}$)

Tight bounds

- Suppose that $\underline{\mathbf{x}} \in X$ is a feasible solution to ILP (max-problem) and that $\bar{\mathbf{x}}$ solves a relaxation of ILP
- Then
$$\underline{z} := \mathbf{c}^{\mathrm{T}}\underline{\mathbf{x}} \leq z^* \leq \mathbf{c}^{\mathrm{T}}\bar{\mathbf{x}} =: \bar{z}$$
- $\bar{z}$ is an *optimistic* estimate of z^*
- $\underline{z}$ is a *pessimistic* estimate of z^*
- If $\bar{z} - \underline{z} \leq \varepsilon$ then the value of the solution candidate $\underline{\mathbf{x}}$ is at most ε from the optimal value z^*
- Efficient solution methods for ILP combine relaxation and heuristic methods to get tight bounds (small $\varepsilon \geq 0$)

Good and ideal formulations



Ideal since all extreme points are integral

Linear program has integer extreme points

Cutting plane algorithms (iterativley tighter relaxations)

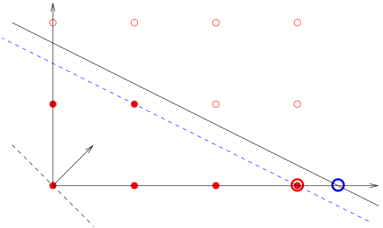
- Solve the linear programming (continuous) relaxation
  - If the solution is integer, then an optimal solution is found
  - Otherwise, find a *cut*, i.e. a linear constraint that cuts off the fractional solution, *but none of the integer solutions*
- The cut should also pass through at least one integer point ($\Rightarrow$ faster convergence)
- Add cuts to the current linear program and resolve until an integer solution is found
- *Remark:* An inequality in higher dimensions defines a *hyper-plane*; therefore the name cutting *plane*

Cutting planes: A very small example

- Consider the following ILP:

$$\max\{x_1 + x_2 : 2x_1 + 4x_2 \leq 7, x_1, x_2 \geq 0 \text{ and integer}\}$$

- ILP solution: $z = 3$, $\mathbf{x} = (3, 0)$
- LP solution (continuous relaxation): $z = 3.5$, $\mathbf{x} = (3.5, 0)$
- Generate a simple cut: "Divide the constraint" by 2: $x_1 + 2x_2 \leq 3.5 \Rightarrow x_1 + 2x_2 \leq 3$
- Adding this cut to the continuous relaxation yields the optimal ILP solution

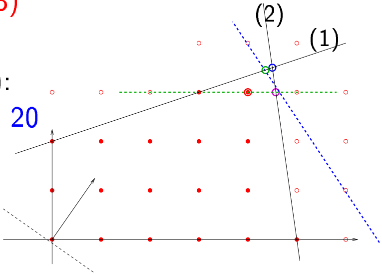


Cutting planes: An example using valid inequalities (VI)

- Consider the ILP

$$\begin{array}{llll} \max & 7x_1 + 10x_2 & & \\ \text{subject to} & -x_1 + 3x_2 & \leq 6 & (1) \\ & 7x_1 + x_2 & \leq 35 & (2) \\ & x_1, x_2 & \geq 0, \text{ integer} & \end{array}$$

- LP optimum: $z = 66.5$, $\mathbf{x} = (4.5, 3.5)$
- ILP optimum: $z = 58$, $\mathbf{x} = (4, 3)$
- Generate a VI by "adding" the two constraints (1) and (2): $6x_1 + 4x_2 \leq 41 \Rightarrow 3x_1 + 2x_2 \leq 20$ $\Rightarrow \mathbf{x} = (4.36, 3.45)$
- Generate a VI by "7·(1)+(2)": $22x_2 <= 77 \Rightarrow x_2 \leq 3$ $\Rightarrow \mathbf{x} = (4.57, 3)$



Cutting plane algorithms

- Problem: It may be necessary to generate MANY cuts
- General methods: E.g., Chvatal-Gomory cuts (combine constraints, make beneficial roundings of LHS and RHS)
- Pure cutting plane algorithms are usually less efficient than branch–&–bound
- In commercial solvers (e.g. CPLEX), cuts are used to help (presolve) the branch–&–bound algorithm
- If the problem has a specific structure, as e.g. TSP and set covering, problem specific classes of cuts are used

Lagrangian relaxation (yields optimistic estimates)

- Consider a maximization integer linear program (ILP):

$$
\begin{array}{llrcll}
[\text{ILP}] & z^* = & \max & \mathbf{c}^{\mathrm{T}}\mathbf{x} & & \\
& & \text{subject to} & \mathbf{A}\mathbf{x} & \leq & \mathbf{b} & (1) \\
& & & \mathbf{D}\mathbf{x} & \leq & \mathbf{d} & (2) \\
& & & \mathbf{x} & \geq & \mathbf{0} \text{ and integer} &
\end{array}
$$

- Assume that the constraints (1) are complicating (subtour eliminating constraints for TSP, e.g.)
- Define the set $X = \{\mathbf{x} \in Z_+^n \,|\, \mathbf{D}\mathbf{x} \leq \mathbf{d}\}$
- Remove the constraints (1) and add them—with penalty parameters $\mathbf{v}$—to the objective function

$$q(\mathbf{v}) = \max_{\mathbf{x} \in X} \left\{ \mathbf{c}^{\mathrm{T}}\mathbf{x} + \mathbf{v}^{\mathrm{T}}(\mathbf{b} - \mathbf{A}\mathbf{x}) \right\} \qquad (3)$$

Weak duality of Lagrangian relaxations

Theorem: For any $\mathbf{v} \geq \mathbf{0}$ it holds that $q(\mathbf{v}) \geq z^*$.

Proof: Let $\bar{\mathbf{x}}$ be feasible in [ILP] $\Rightarrow \bar{\mathbf{x}} \in X$ and $\mathbf{A}\bar{\mathbf{x}} \leq \mathbf{b}$. It then holds that

$$q(\mathbf{v}) = \max_{\mathbf{x} \in X} \{\mathbf{c}^{\mathrm{T}}\mathbf{x} + \mathbf{v}^{\mathrm{T}}(\mathbf{b} - \mathbf{A}\mathbf{x})\} \geq \mathbf{c}^{\mathrm{T}}\bar{\mathbf{x}} + \mathbf{v}^{\mathrm{T}}(\mathbf{b} - \mathbf{A}\bar{\mathbf{x}}) \geq \mathbf{c}^{\mathrm{T}}\bar{\mathbf{x}}.$$

Since an optimal solution $\mathbf{x}^*$ to [ILP] is also feasible, it holds that

$$q(\mathbf{v}) \geq \mathbf{c}^{\mathrm{T}}\mathbf{x}^* = z^*. \qquad \square$$

$\Rightarrow$ $q(\mathbf{v})$ is an *upper bound* on the optimal value z^* for any $\mathbf{v} \geq \mathbf{0}$

- The best upper bound is given by

$$q^* = \min_{\mathbf{v} \geq \mathbf{0}} q(\mathbf{v}) = \min_{\mathbf{v} \geq \mathbf{0}} \left\{ \max_{\mathbf{x} \in X} \{\mathbf{c}^{\mathrm{T}}\mathbf{x} + \mathbf{v}^{\mathrm{T}}(\mathbf{b} - \mathbf{A}\mathbf{x})\} \right\}$$

Tractable integer Lagrangian relaxations

- Special algorithms for minimizing the Lagrangian dual function q exist
- q is always convex but typically nondifferentiable
- For each value of $\mathbf{v}$ chosen, a *subproblem* (3) must be solved
- For general ILP:s there is typically a non-zero duality gap: $q^* > z^*$
- The Lagrangian relaxation bound is never worse that the linear programming relaxation bound, i.e. $z^{\mathrm{LP}} \geq q^* \geq z^*$
- If the set X has the integrality property (i.e., X^{LP} possesses integral extreme points) then $z^{\mathrm{LP}} = q^*$
- Choose the constraints ($\mathbf{Ax} \leq \mathbf{b}$) to dualize such that the relaxed problem (3) is computationally tractable but still does *not* possess the integrality property

Branch–&–Bound algorithms (B&B)

$$[\text{ILP}] \qquad z^* = \max_{\mathbf{x} \in X} \mathbf{c}^{\mathrm{T}}\mathbf{x}, \qquad X \subset Z^n$$

- A general principle for finding *optimal* solutions to optimization problems with integrality requirements
- Can be adopted to different types of models
- Can be combined with other (e.g. heuristic) algorithms
- Also called implicit enumeration and tree search
- *Idea:* Enumerate all feasible solutions by a successive partitioning of X into a family of subsets
- Enumeration organized in a tree using graph search; it is made implicit by utilizing approximations of z^* from relaxations of [ILP] for cutting off branches of the tree
- The worst case-complexity for B&B is exponential

Branch–&–bound: Main concepts

- Relaxation: a simplification of [ILP] in which some constraints are removed
  - Purpose: to get simple (polynomially solvable) (node) subproblems, and optimistic approximations of z^*.
- Branching strategy: rules for partitioning a subset of X
  - Purpose: exclude the solution to a relaxation if it is not feasible in [ILP]
- Tree search strategy: defines the order in which the nodes in the B&B tree are created and searched
  - Purpose: quickly find good feasible solutions; limit the size of the tree
- Node cutting criteria: rules for deciding when a subset should not be further partitioned
  - Purpose: avoid searching parts of the tree that cannot contain an optimal solution

B&B

- Relaxations: remove integrality requirements, remove/Lagrangean relax complicating (linear) constraints
- Branching: should correspond to a partitioning of the feasible set
- Tree search: depth–first, bredth–first, best–first, ...
- Cut off a node if the corresponding node subproblem has
  - no feasible solution, or
  - an optimal solution that is feasible in [ILP], or
  - an optimal objective value that is worse (lower) than that of any known feasible solution

B&B: An Example

Solve the following ILP example using the branch–&–bound algorithm

$$\begin{aligned} \max \quad & 5x_1 + 4x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 5 \\ & 10x_1 + 6x_2 \leq 45 \end{aligned}$$

LP-optimum is $z = 23.75$, $x_1 = 3.75$ and $x_2 = 1.25$.

Local search—generating feasible solutions (pessimistic estimates of z^*)

Consider a maximization problem:

$$\max_{\mathbf{x} \in X} \quad \mathbf{c}^{\mathrm{T}} \mathbf{x}$$

0. Initialization: Choose a feasible solution $\mathbf{x}^0$. Let $t = 0$.
1. Find all feasible points in a neighbourhood $N(\mathbf{x}^k)$ of $\mathbf{x}^k$
2. If $\mathbf{c}^{\mathrm{T}}\mathbf{x} \leq \mathbf{c}^{\mathrm{T}}\mathbf{x}^k$ for all $\mathbf{x} \in X \cap N(\mathbf{x}^k) \Rightarrow$ Stop, $\mathbf{x}^k$ is a local optimum
3. Choose $\mathbf{x}^{k+1} \in X \cap N(\mathbf{x}^k)$ such that $\mathbf{c}^{\mathrm{T}}\mathbf{x}^{k+1} > \mathbf{c}^{\mathrm{T}}\mathbf{x}^k$
4. Let $k := k + 1$ and go to step 1

More about local search heuristics

- Starting feasible solution from constructive heuristic
- Definition of neighbourhood is model specific
- Finds a *local* optimal solution
- *No guarantee* to find global optimal solutions
- Extensions (e.g. tabu search): Temporarily allow worse solutions to move away from a local optimum
- Larger neighbourhoods yield better local optima, but takes more computational time