MVE165/MMG630, Applied Optimization
Lecture 1
Introduction; course map; modelling optimization applications; linear, nonlinear, and integer programs; graphic solution; software solvers

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Staff and homepage

Staff

- **Examiner/lecturer:** Ann-Brith Strömberg (anstr@chalmers.se, room L2087)
- **Guest lecturers:**
  - Fredrik Hedenus (Energy and Environment),
  - Michael Patriksson (Mathematical Sciences),
  - Caroline Olsson (Radiation Physics, Clinical Sciences), and
  - Elin Svensson (Energy and Environment)

Course homepage

`http://www.math.chalmers.se/Math/Grundutb/CTH/mve165/0910/`

- Contains details, information on assignmnents and exercises, deadlines, lecture notes, etc
- Will be updated with new information every week during the course

Course contents and organization

Contents

- Applications of optimization
- Mathematical modelling
- Solution techniques – algorithms
- Software solvers

Organization

- Lectures – mathematical optimization theory
- Exercises – use solvers, oral examination (or report) of #2
- Guest lectures – applications of optimization
- Assignments – modelling, use solvers, written reports, opposition & oral presentations
- Assignment work should be done in groups of two persons

Literature

- Main course book:
  - English version: Optimization (2010)
  - Swedish version: Optimeringslära (2008)

  by J. Lundgren, M. Rönnqvist, and P. Värbrand. Studentlitteratur.
- Exercises:
  - English version: Optimization Exercises (2010)
  - Swedish version: Optimeringslära Övningsbok (2008)

  by M. Henningsson, J. Lundgren, M. Rönnqvist, and P. Värbrand. Studentlitteratur.
- Cremona/Studentlitteratur/Adlibris/...
- Hand-outs

Examination

- A correctly solved computer Exercise #2 (oral examination or written report)
- Written reports of three assignments (1, 2, and 3a or 3b)
- A written opposition to Assignment 2
- An oral presentation of Assignment 3a or 3b
- To be able to receive a grade higher than 3 or G, the written reports and opposition as well as the oral presentation must be of high quality. Students aiming at grade 4, 5, or VG must also pass an oral exam

Overview of the lectures

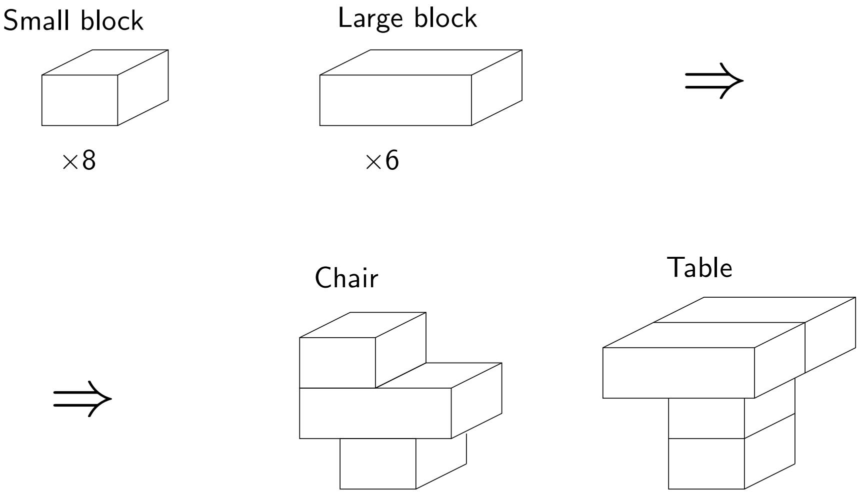
- Linear programming, modelling, theory, solution methods, sensitivity analysis
- Optimization models that can be described as flows in networks, solution methods
- Discrete optimization models and solution methods
- Non-linear programming models, with and without constraints, solution methods
- Multiple objective optimization
- Optimization under uncertainty
- Mixtures of the above

Optimization

“Do something as good as possible”

- **Something:** Which are the decision alternatives?
- **Possible:** What restrictions are there?
- **Good:** What is a relevant optimization criterion?

A manufacturing example:
Produce tables and chairs from two types of blocks



A manufacturing example, continued

- A chair is assembled from one large and two small blocks
- A table is assembled from two blocks of each size
- Only 6 large and 8 small blocks are avaliable
- A table is sold at a revenue of 1600:-
- A chair is sold at a revenue of 1000:-
- Assume that all items produced can be sold and determine an optimal production plan

A mathematical optimization model

Something: Which are the decision alternatives? $\Rightarrow$ *Variables*

$$\begin{aligned} x_1 &= \text{number of tables produced and sold} \\ x_2 &= \text{number of chairs produced and sold} \end{aligned}$$

Possible: What restrictions are there? $\Rightarrow$ *Constraints*

$$\begin{array}{rcrcll} 2x_1 & + & x_2 & \leq & 6 & \text{(6 large blocks)} \\ 2x_1 & + & 2x_2 & \leq & 8 & \text{(8 small blocks)} \\ & & x_1,\ x_2 & \geq & 0 & \text{(physical restrictions)} \\ & & (x_1,\ x_2 & \multicolumn{2}{l}{\text{integral})} & \text{(physical restrictions)} \end{array}$$

Good: What is a relevant optimization criterion? $\Rightarrow$ *Objective function*

$$\text{maximize} \quad z = 1600x_1 + 1000x_2 \qquad (z = \text{total revenue})$$

Solve the model using LEGO!

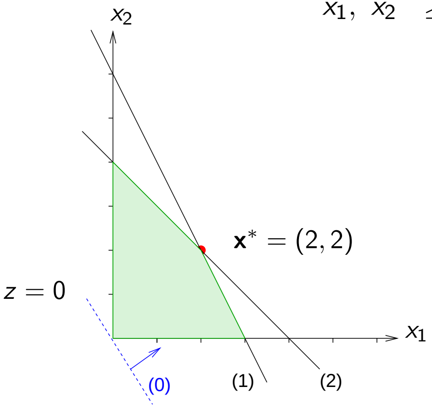
- Start at no production: $x_1 = x_2 = 0$
  Use the "best marginal profit" to choose the item to produce
  - x_1 has the highest marginal profit (1600:-/table)
    $\Rightarrow$ produce as many tables as possible
  - At $x_1 = 3$: no more large blocks left

- The marginal value of x_2 is now 200:- since taking apart one table (-1600:-) yields two chairs ($+2000$:-) $\Rightarrow$ 400:-/2 chairs
  - Increase x_2 maximally $\Rightarrow$ decrease x_1
  - At $x_1 = x_2 = 2$: no more small blocks

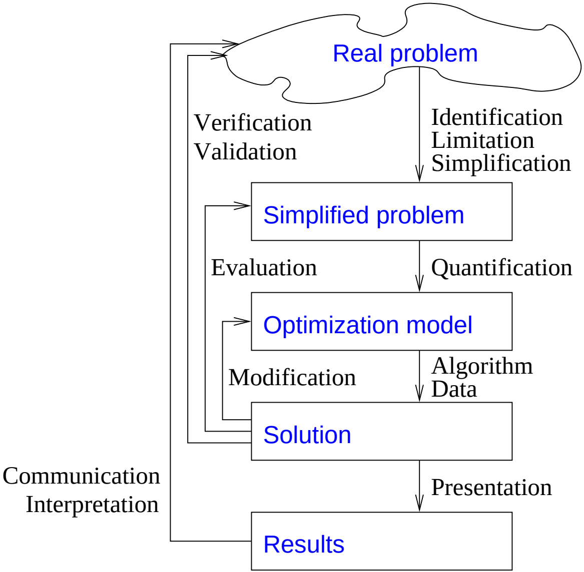
- The marginal value of x_1 is negative (to build one more table one has to take apart two chairs $\Rightarrow$ -400:-)
  The marginal value of x_2 is -600:- (to build one more chair one table must be taken apart)
  $\Longrightarrow$ Optimal solution: $x_1 = x_2 = 2$

Geometric solution of the model

$$
\begin{array}{lrcrcrcrr}
\text{maximize} & z & = & 1600x_1 & + & 1000x_2 & & & (0) \\
\text{subject to} & & & 2x_1 & + & x_2 & \leq & 6 & (1) \\
 & & & 2x_1 & + & 2x_2 & \leq & 8 & (2) \\
 & & & & & x_1,\ x_2 & \geq & 0 &
\end{array}
$$



Operations research—more than just mathematics



Concepts

- **Feasible solution:** A solution that satisfies all constraints
- **Optimal solution:** A solution that is feasible AND whose objective function value is as good as that of every feasible solution
- **Sensitivity analysis:** How does the optimal solution depend on the input parameters?
- **Tractability:** Can the the model be solved in reasonable time?
- **Validity:** Does the conclusions drawn from the solution hold for the REAL problem
- **Operations research:** Always a tradeoff between validity of the model and its tractability to analyse

More concepts

- **Heuristic or approximate solution:** A solution that is feasible, but not guaranteed to be optimal. Quality measures can be computed
- **Deterministic optimization model:** All parameter values are assumed to be known for sure
- **Stochastic optimization model:** Involves quantities known only by probability (optimization under uncertainty)
- **Multiple objective optimization:** Typically, NO feasible solution exists that is optimal for ALL objectives. Search for Pareto optimal solutions

Optimization modelling: A production–inventory example

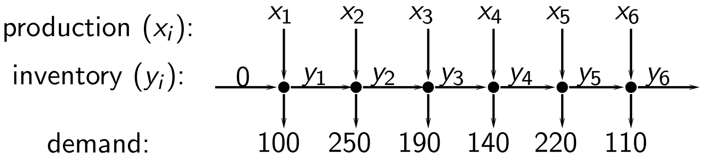
- Commission: Deliver windows over a six-month period
- Demand during the respective months: 100, 250, 190, 140, 220 & 110 units
- Production cost per unit (window): 50 €,45 €, 55 €, 48 €, 52 € & 50 €
- Store a manufactured window from one month to the next at 8 €
- Requirement: Meet the demand and minimize the costs
- Find an optimal production schedule

Define the decision variables

$$x_i \quad = \quad \text{number of units produced in month } i = 1, \ldots, 6$$

$$y_i \quad = \quad \text{units left in the inventory at the end of month } i = 1, \ldots, 6$$

- The "flow" of windows can be illustrated as:



Define the limitations/constraints

- Each month:

initial inventory + production − ending inventory = demand

$$
\begin{array}{ccccccccl}
0 & + & x_1 & - & y_1 & = & 100 & & \\
y_1 & + & x_2 & - & y_2 & = & 250 & & \\
y_2 & + & x_3 & - & y_3 & = & 190 & & \\
y_3 & + & x_4 & - & y_4 & = & 140 & & \\
y_4 & + & x_5 & - & y_5 & = & 220 & & \\
y_5 & + & x_6 & - & y_6 & = & 110 & & \\
 & & x_i & , & y_i & \geq & 0, & & i = 1, \ldots, 6
\end{array}
$$

Objective function: minimize the costs for production and storage

- Production cost (€):

$$50\ x_1 + 45\ x_2 + 55\ x_3 + 48\ x_4 + 52\ x_5 + 50\ x_6$$

- Inventory cost (€):

$$8\ (y_1 + y_2 + y_3 + y_4 + y_5 + y_6)$$

- Objective:

$$\begin{aligned} \text{minimize} \quad & 50x_1 + 45x_2 + 55x_3 + 48x_4 + 52x_5 + 50x_6 \\ & +8(y_1 + y_2 + y_3 + y_4 + y_5 + y_6) \end{aligned}$$

A complete (general) optimization model

$$
\begin{array}{llcll}
\text{minimize} & \displaystyle\sum_{i=1}^{6} c_i x_i + 8 \sum_{i=1}^{6} y_i, & & & \\
\text{subject to} & y_{i-1} + x_i - y_i & = & d_i, & i = 1, \dots, 6, \\
& y_0 & = & 0, & \\
& x_i, y_i & \geq & 0, & i = 1, \dots, 6,
\end{array}
$$

The vector of demand:

$$d = (d_i)_{i=1}^{6} = (100, 250, 190, 140, 220, 110)$$

The vector of production costs:

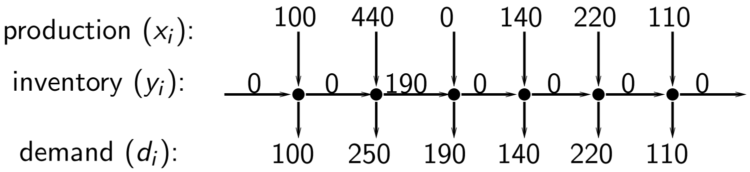
$$c = (c_i)_{i=1}^{6} = (50, 45, 55, 48, 52, 50)$$

.

An optimal solution—optimal production schedule

Optimal production each month:
$x = (x_i)_{i=1}^{6} = (100, 440, 0, 140, 220, 110)$

Optimal inventory each month:
$y = (y_i)_{i=0}^{6} = (0, 0, 190, 0, 0, 0, 0)$



The minimal total cost is 49980 €

Mathematical optimization models

$$\left[\begin{array}{lll} \text{minimize or maximize} & f(x_1,\dots,x_n) & \\ \text{subject to} & g_i(x_1,\dots,x_n) \left\{\begin{array}{c}\le\\=\\\ge\end{array}\right\} b_i, & i=1,\dots,m \end{array}\right]$$

- $x_1,\dots,x_n$ are the decision variables
- f and $g_1,\dots,g_m$ are given functions of the decision variables
- $b_1,\dots,b_m$ are specified constant parameters
- The functions can be nonlinear, e.g. quadratic, exponential, logarithmic, non-analytic, ...
- In general, linear forms are more tractable than non-linear

Linear programming models

- The production inventory model is a linear program (LP), i.e., all relations are described by linear forms
- A general linear program:

$$\left[\begin{array}{llccl} \text{min or max} & c_1x_1 + \ldots + c_nx_n & & & \\ \text{subject to} & a_{i1}x_1 + \ldots + a_{in}x_n & \left\{\begin{array}{c}\leq\\=\\\geq\end{array}\right\} & b_i, & i = 1, \ldots, m \\ & x_j & \geq & 0, & j = 1, \ldots, n \end{array}\right]$$

- The non-negativity constraints on x_j, $j = 1, \ldots, n$ are not necessary, but usually assumed (reformulation always possible)

Discrete/integer/binary modelling

- A variable is called *discrete* if it can take only a countable set of values, e.g.,
  - Continuous variable: $x \in [0, 8]$ or $0 \leq x \leq 8$
  - Discrete variable: $x \in \{0, 4.4, 5.2, 8.0\}$
  - *Integer* variable: $x \in \{0, 1, 4, 5, 8\}$
- A *binary* variable can only take the values 0 or 1, i.e., all or nothing

  E.g., a wind-mill can produce electricity only if it is built
  - Let $y = 1$ if the mill is built, otherwise $y = 0$
  - Capacity of a mill: C
  - Production $x \leq Cy$ (also limited by wind force etc.)
- In general, models with only continuous variables are more tractable than models with integrality/discrete requirements on the variables, but exceptions exist!

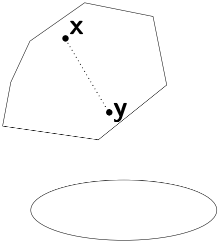
Convex sets

- A set S is convex if, for any elements $\mathbf{x}, \mathbf{y} \in S$ it holds that

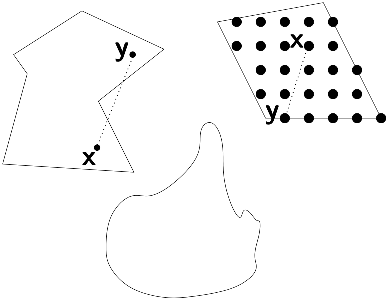
$$\alpha \mathbf{x} + (1 - \alpha)\mathbf{y} \in S \text{ for all } 0 \leq \alpha \leq 1$$

- Examples:

Convex sets



Non-convex sets



$\Rightarrow$ Intersections of linear (in)equalities $\Rightarrow$ convex sets

Convex and concave functions

- A function f is convex on the set S if, for any elements $\mathbf{x}, \mathbf{y} \in S$ it holds that

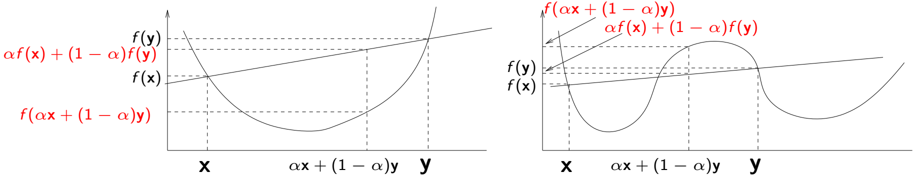
$$f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \leq \alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y}) \text{ for all } 0 \leq \alpha \leq 1$$

- A function f is concave on the set S if, for any elements $\mathbf{x}, \mathbf{y} \in S$ it holds that

$$f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \geq \alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y}) \text{ for all } 0 \leq \alpha \leq 1$$

$\Rightarrow$ Linear functions are convex (and concave)

Convex function | Non-convex function



Global solutions of convex and linear programs

- Let $\mathbf{x}^*$ be a *local* minimizer of a *convex function* over a *convex set*. Then $\mathbf{x}^*$ is also a *global* minimizer.

$\Rightarrow$ Every local optimum of a linear program is a global optimum

- If a linear program has any optimal solutions, at least one optimal solution is at an extreme point of the feasible set

$\Rightarrow$ Search for optimal extreme point(s)

- Next lecture: Linear programs and the simplex method