MVE165/MMG630, Applied Optimization
Lecture 7
Integer linear programming

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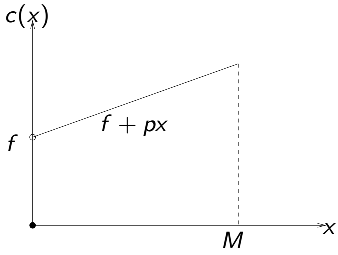
Modelling with integer variables (Ch. 13.1)

- **Variables**
  - *Linear programming* (LP) uses continuous variables: $x_{ij} \geq 0$
  - *Integer linear programming* (ILP) use also *integer*, *binary*, and *discrete* variables
  - If both continuous and integer variables are used in a program, it is called a *mixed integer (linear) program (MILP)*
- **Constraints**
  - In an ILP (or MILP) it is possible to model linear constraints, but also logical relations as, e.g. if–then and either–or
  - This is done by introducing additional binary variables and additional constraints

Mixed integer modelling—fixed charges

- Send a truck $\Rightarrow$ Start–up cost $f > 0$
- Load bread loafs $\Rightarrow$ cost $p > 0$ per loaf
- $x = \#$ bread loafs to transport from bakery to store

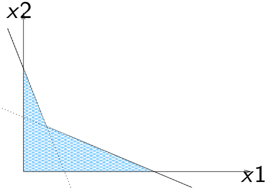


- Cost function $c(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ f + px & \text{if} \quad 0 < x \leq M \end{cases}$
- The function $c : \Re_+ \mapsto \Re_+$ is *nonlinear* and *discontinuos*

Integer linear programming modelling—fixed charges

- Let $y = \#$ trucks to send (here y equals 0 or 1)
- Replace $c(x)$ by $fy + px$
- Constraints: $0 \leq x \leq My$ and $y \in \{0, 1\}$
- New model: $\left[\begin{array}{lrcl} & \min fy + px & & \\ \text{s.t.} & x - My & \leq & 0 \\ & x & \geq & 0 \\ & y & \in & \{0,1\} \end{array} \right]$
  - $y = 0 \quad \Rightarrow \quad x = 0 \quad \Rightarrow \quad fy + px = 0$
  - $y = 1 \quad \Rightarrow \quad x \leq M \quad \Rightarrow \quad fy + px = f + px$
  - $x > 0 \quad \Rightarrow \quad y = 1 \quad \Rightarrow \quad fy + px = f + px$
  - $x = 0 \quad \not\Rightarrow \quad y = 0$ But: Minimization will push y to zero!

Discrete alternatives



- Suppose:
  either $x_1 + 2x_2 \leq 4$ **or** $5x_1 + 3x_2 \leq 10$,
  and $x_1, x_2 \geq 0$ must hold
- **Not** a convex set
- Let $M \gg 1$ and define $y \in \{0, 1\}$

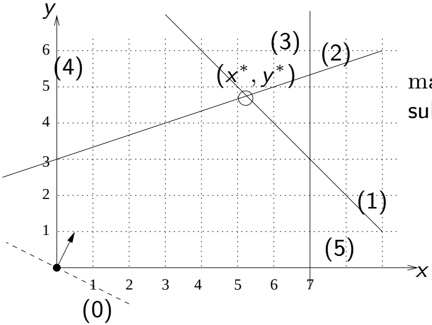
$\Rightarrow$ New constraint set:
$$\begin{bmatrix} x_1 + 2x_2 & -My & \leq & 4 \\ 5x_1 + 3x_2 & -M(1-y) & \leq & 10 \\ & y & \in & \{0,1\} \\ x_1, x_2 & & \geq & 0 \end{bmatrix}$$

- $y = \begin{cases} 0 & \Rightarrow \quad x_1 + 2x_2 \leq 4 \quad \text{must hold} \\ 1 & \Rightarrow \quad 5x_1 + 3x_2 \leq 10 \quad \text{must hold} \end{cases}$

Exercises: Homework

1. Suppose that you are interested in choosing from a set of investments $\{1, \ldots, 7\}$ using $0-1$ variables. Model the following constraints.
    1.1 You cannot invest in all of them
    1.2 You must choose at least one of them
    1.3 Investment 1 cannot be chosen if investment 3 is chosen
    1.4 Investment 4 can be chosen only if investment 2 is also chosen
    1.5 You must choose either both investment 1 and 5 or neither
    1.6 You must choose either at least one of the investments 1, 2 and 3 or at least two investments from 2, 4, 5 and 6
2. Formulate the following as mixed integer progams
    2.1 $u = \min\{x_1, x_2\}$, assuming that $0 \leq x_j \leq C$ for $j = 1, 2$
    2.2 $v = |x_1 - x_2|$ with $0 \leq x_j \leq C$ for $j = 1, 2$
    2.3 The set $X \setminus \{x^*\}$ where $X = \{x \in Z^n | Ax \leq b\}$ and $x^* \in X$

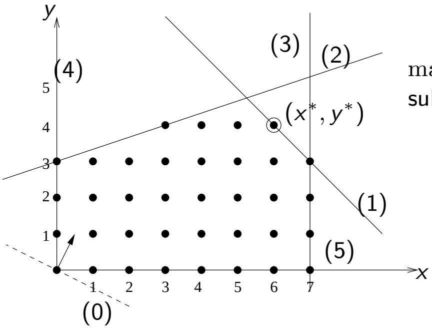
Linear programming: A small example



$$
\begin{array}{lrcrcrl}
\text{maximize} & x & + & 2y & & & (0) \\
\text{subject to} & x & + & y & \leq & 10 & (1) \\
 & -x & + & 3y & \leq & 9 & (2) \\
 & x & & & \leq & 7 & (3) \\
 & & & x, y & \geq & 0 & (4,5)
\end{array}
$$

- Optimal solution: $(x^*, y^*) = (5\frac{1}{4}, 4\frac{3}{4})$
- Optimal objective value: $14\frac{3}{4}$

Integer linear programming: A small example



$$
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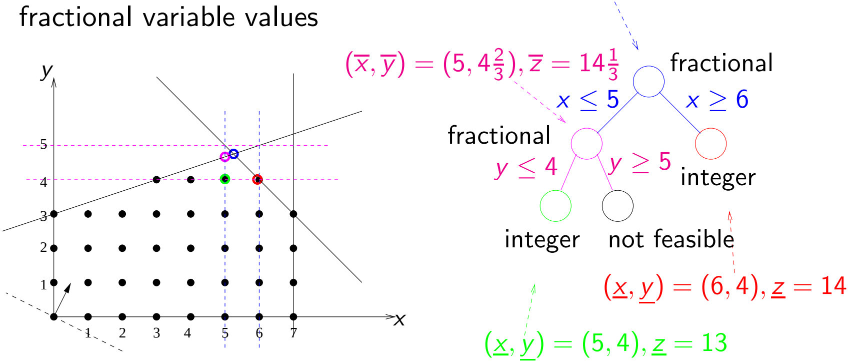
\text{subject to} & x & + & y & \leq & 10 & (1) \\
 & -x & + & 3y & \leq & 9 & (2) \\
 & x & & & \leq & 7 & (3) \\
 & & & x, y & \geq & 0 & (4,5) \\
 & & & x, y & \text{integer} & &
\end{array}
$$

- What if the variables are forced to be integral?
- Optimal solution: $(x^*, y^*) = (6, 4)$
- Optimal objective value: $14 < 14\frac{3}{4}$
- The optimal value decreases (possibly constant) when the variables are restricted to have integral values

ILP: Solution by the branch–and–bound algorithm (e.g., Cplex, XpressMP, or GLPK) (Ch. 15.1–15.2)

- Relax integrality requirements $\Rightarrow$ linear, continuous problem $\Rightarrow$ $(\overline{x}, \overline{y}) = (5\frac{1}{4}, 4\frac{3}{4}), \overline{z} = 14\frac{3}{4}$
- Search tree: branch over fractional variable values



The knapsack problem—budget constraints (Ch. 13.2)

- Select an optimal collection of objects or investments or projects ...
  - c_j = benefit of choosing object j, $j = 1, \ldots, n$
- Limits on the budget
  - a_j = cost of object j, $j = 1, \ldots, n$
  - b = total budget
- Variables: $x_j = \begin{cases} 1, & \text{if object } j \text{ is chosen,} \\ 0, & \text{otherwise.} \end{cases} \quad j = 1, \ldots, n$
- Objective function: $\boxed{\max \sum_{j=1}^{n} c_j x_j}$
- Budget constraint: $\boxed{\sum_{j=1}^{n} a_j x_j \leq b}$
- Binary variables: $\boxed{x_j \in \{0, 1\},\ j = 1, \ldots, n}$

Computational complexity (Ch. 2.6)

- A small knapsack instance

$$\begin{array}{lrcl} z_1^* = \max & 213x_1 + 1928x_2 + 11111x_3 + 2345x_4 + 9123x_5 & & \\ \text{subject to} & 12223x_1 + 12224x_2 + 36674x_3 + 61119x_4 + 85569x_5 & \leq & 89\ 643\ 482 \\ & x_1, \ldots, x_5 & \geq & 0, \text{integer} \end{array}$$

- Optimal solution $\mathbf{x}^* = (0, 1, 2444, 0, 0)$, $z_1^* = 27\ 157\ 212$
- Cplex finds this solution in 0.015 seconds
- The equality version

$$\begin{array}{lrcl} z_2^* = \max & 213x_1 + 1928x_2 + 11111x_3 + 2345x_4 + 9123x_5 & & \\ \text{subject to} & 12223x_1 + 12224x_2 + 36674x_3 + 61119x_4 + 85569x_5 & = & 89\ 643\ 482 \\ & x_1, \ldots, x_5 & \geq & 0, \text{integer} \end{array}$$

- Optimal solution $\mathbf{x}^* = (7334, 0, 0, 0, 0)$, $z_2^* = 1\ 562\ 142$
- Cplex computations interrupted after 1700 sec. ($\approx \frac{1}{2}$ hour)
  - No integer solution found
  - Best upper bound found: 25 821 000
  - 55 863 802 branch–and–bound nodes visited
  - Only *one* feasible solution exists!

Computational complexity

- Mathematical insight yields successful algorithms
- Example: Assignment problem: Assign n persons to n jobs.
- $\#$ feasible solutions: $n!$ $\Rightarrow$ Combinatorial explosion
- An algorithm $\exists$ that solves this problem in time $\mathcal{O}(n^4) \propto n^4$
- Binary knapsack: $\mathcal{O}(2^n)$
- Complete enumeration of all solutions is *not* efficient

n	2	5	8	10	100	1000
$n!$	2	120	40000	3600000	$9.3 \cdot 10^{157}$	$4.0 \cdot 10^{2567}$
2^n	4	32	256	1024	$1.3 \cdot 10^{30}$	$1.1 \cdot 10^{301}$
n^4	16	625	4100	10000	100000000	$1.0 \cdot 10^{12}$
($n \log n$	0.6	3.5	7.2	10	200	3000)

- (Continuous knapsack (sorting of $\frac{c_j}{a_j}$): $\mathcal{O}(n \log n)$)

The set covering problem (Ch. 13.8)

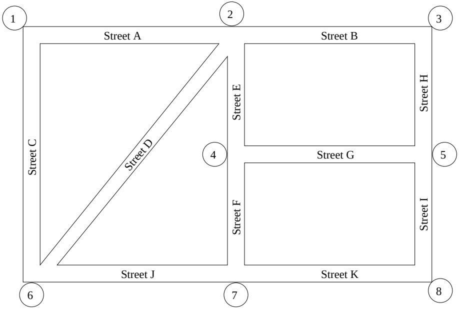
- A number (n) of items and a cost for each item
- A number (m) of subsets of the n items
- Find a selection of the items such that each subset contains at least one selected item and such that the total cost for the selected items is minimized
- Mathematical formulation:

$$\begin{array}{lrcl} \min & \mathbf{c}^{\mathrm{T}}\mathbf{x} & & \\ \text{subject to} & \mathbf{A}\mathbf{x} & \geq & \mathbf{1} \\ & \mathbf{x} & & \text{binary} \end{array}$$

- $\mathbf{c} \in \Re^n$ and $\mathbf{1} = (1, \ldots, 1)^{\mathrm{T}} \in \Re^m$ are constant vectors
- $\mathbf{A} \in \Re^{m \times n}$ is a matrix with entries $a_{ij} \in \{0, 1\}$
- $\mathbf{x} \in \Re^n$ is the vector of variables
- Related models: *set partitioning* ($\mathbf{A}\mathbf{x} = \mathbf{1}$), *set packing* ($\mathbf{A}\mathbf{x} \leq \mathbf{1}$)

Example: Installing security telephones

- The road administration wants to install emergency telephones such that each street has access to at least one phone
- It is logical to place the phones at street crossings
- Each crossing has an installation cost: $\mathbf{c} = (2, 2, 3, 4, 3, 2, 2, 1)$
- Find the cheapest selection of crossings to provide all streets with phones

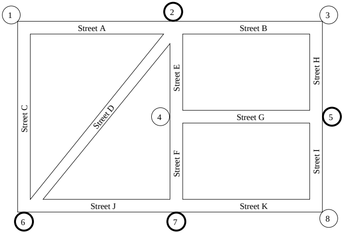


- Define variables and constraints

Installing security telephones: Mathematical model

- Binary variables for each crossing: $x_j = 1$ if a phone is installed at j, $x_j = 0$ otherwise.
- For each street, introduce a constraint saying that a phone should be placed at—at least—one of its crossings:

  A: $x_1 + x_2 \geq 1$, B: $x_2 + x_3 \geq 1$,
  C: $x_1 + x_6 \geq 1$, D: $x_2 + x_6 \geq 1$,
  E: $x_2 + x_4 \geq 1$, F: $x_4 + x_7 \geq 1$,
  G: $x_4 + x_5 \geq 1$, H: $x_3 + x_5 \geq 1$,
  I: $x_5 + x_8 \geq 1$, J: $x_6 + x_7 \geq 1$,
  K: $x_7 + x_8 \geq 1$



- Objective function:
  $$\min \quad 2x_1 + 2x_2 + 3x_3 + 4x_4 + 3x_5 + 2x_6 + 2x_7 + x_8$$
- An optimal solution: $x_2 = x_5 = x_6 = x_7 = 1$, $x_1 = x_3 = x_4 = x_8 = 0$. Objective value: 9.

More modelling examples (Ch. 13.3)

- Given three telephone companies A, B and, C which charge a fixed start-up price of 16, 25 and, 18, respectively
- For each minute of call-time A, B, and, C charge 0.25, 0.21 and, 0.22
- We want to phone 200 minutes. Which company should we choose?
- x_i = number of minutes called by $i \in \{A, B, C\}$
- Binary variables $y_i = 1$ if $x_i > 0$, $y_i = 0$ otherwise (pay start-up price only if calls are made with company i)
- Mathematical model

$$
\begin{array}{llcl}
\min & 0.25x_1 + 0.21x_2 + 0.22x_3 & + & 16y_1 + 25y_2 + 18y_3 \\
\text{subject to} & x_1 + x_2 + x_3 & = & 200 \\
 & 0 \leq x_i & \leq & 200y_i, \quad i = 1, 2, 3 \\
 & y_i & \in & \{0, 1\}, \quad i = 1, 2, 3
\end{array}
$$

More modelling examples (2) (Ch. 13.9)

- We wish to process three jobs on one machine
- Each job j has a processing time p_j, a due date d_j, and a penalty cost c_j if the due date is missed
- How should the jobs be scheduled to minimize the total penalty cost?

Job	Processing time (days)	Due date (days)	Late penalty \$/day
1	5	25	19
2	20	22	12
3	15	35	34

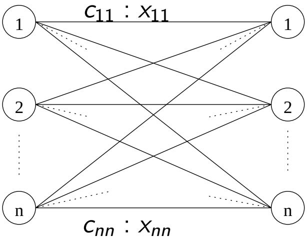
The assignment model (Ch. 13.5)

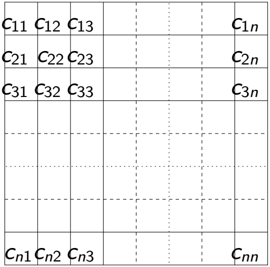
Assign each task to one resource, and each resource to one task

- Linear cost c_{ij} for assigning task i to resource j, $i, j \in \{1, \ldots, n\}$
- Variables: $x_{ij} = \begin{cases} 1, & \text{if task } i \text{ is assigned to resource } j \\ 0, & \text{otherwise} \end{cases}$

$$
\begin{array}{llcll}
\min & \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} & & & \\
\text{subject to} & \displaystyle\sum_{j=1}^{n} x_{ij} & = & 1, & i = 1, \ldots, n \\
 & \displaystyle\sum_{i=1}^{n} x_{ij} & = & 1, & j = 1, \ldots, n \\
 & x_{ij} & \geq & 0, & i, j = 1, \ldots, n
\end{array}
$$

- Choose *one* element from each row and each column

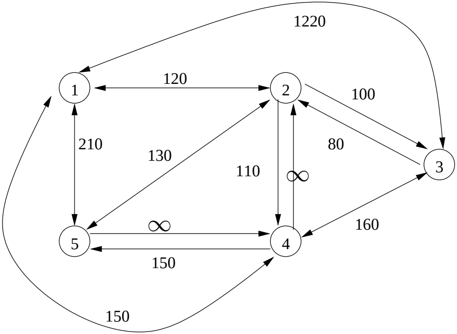




- This integer linear model has integral extreme points, since it can be formulated as a network flow problem
- Therefore, it can be efficiently solved using specialized (network) linear programming techniques
- Even more efficient special purpose (primal–dual–graph-based) algorithms exist

The travelling salesperson problem (TSP) (Ch. 13.10)

- Given n cities and connections between all cities (distances on each connection)
- Find shortest tour that passes through all the cities



- A problem that is very easy to describe and understand but very difficult to solve (combinatorial explosion)
- ∃ different versions of TSP: Euclidean, metric, symmetric, ...

An ILP formulation of the TSP problem

- Let the distance from city i to city j be d_{ij}
- Introduce binary variables x_{ij} for each connection
- Let $V = \{1, \dots, n\}$ denote the set of nodes (cities)

$$
\begin{array}{llcll}
\min & \displaystyle\sum_{i \in V} \sum_{j \in V} d_{ij} x_{ij}, & & & \\
\text{s.t.} & \displaystyle\sum_{j \in V} x_{ij} & = & 1, & i \in V, \qquad (1) \\
& \displaystyle\sum_{i \in V} x_{ij} & = & 1, & j \in V, \qquad (2) \\
& \displaystyle\sum_{i \in U, j \in V \setminus U} x_{ij} & \geq & 1, & \forall U \subset V : 2 \leq |U| \leq |V| - 2, \qquad (3) \\
& x_{ij} & \multicolumn{2}{l}{\text{binary}} & i, j \in V
\end{array}
$$

- Cf. the assignment problem $\boxed{\text{DRAW GRAPH * 2 !}}$
- Enter and leave each city exactly once $\Leftrightarrow$ (1) and (2) $\boxed{\text{DRAW!}}$
- Constraints (3): *subtour elimination* $\boxed{\text{DRAW!}}$
- Alternative formulation of (3): $\boxed{\text{DRAW!}}$
  $\sum_{(i,j) \in U} x_{ij} \leq |U| - 1, \quad \forall U \subset V : 2 \leq |U| \leq |V| - 2$

Solution methods for the TSP Problem

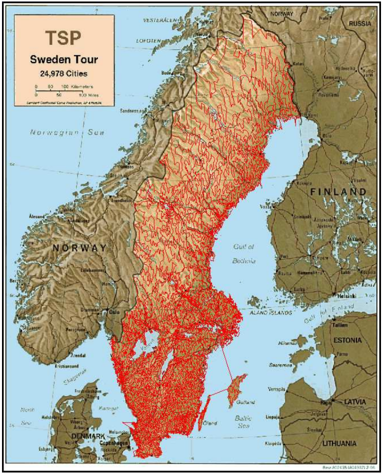
- Branch–&–bound (Ch. 15)
- Cutting plane algorithms (Ch. 14.5)
- Heuristics
  - Constructive heuristics (Ch. 16.3), e.g., Nearest neighbor
  - Local search heuristics (Ch. 16.4), e.g., (2-, 3-interchange)
  - Approximation algorithms (Ch. 16.6): e.g., spanning tree & Christofides heuristic (spanning tree + "some" arcs ...)
  - ...
- ...
- Common difficulty for *all* solution methods for the TSP: *Combinatorial explosion*: # possible tours $\approx n!$

$\Rightarrow$ Very many subtour elimination constraints

The development of TSP solution

- Optimal solutions to TSP's of different sizes found

year	$n =$
1954	49
1962	33
1977	120
1987	532
1987	666
1987	2392
1994	7397
1998	13509
2001	15112
2004	24978



The worlds largest TSP solved so far (2004) ...

- A TSP of 24 978 cities, towns, and villages in Sweden
- Optimal tour: $\approx$ 72 500 km (855597 TSP LIB units)
- The tour of length 855 597 was found in March 2003 (Lin-Kernighan's TSP heuristic)
- It was proven in May 2004 that no shorter tour exists
- The final stages that improved the lower bound from 855 595 up to 855 597 required $\approx$ 8 years of computation time (running in parallel on a network of Linux workstations)

  "Without knowledge of the 855 597 tour we would not have made the decision to carry out this final computation"
- `www.tsp.gatech.edu`
- New record in 2005/06: 85 900 cities in a VLSI application `www.tsp.gatech.edu/pla85900/index.html`

Facility location—a mixed integer linear model (Ch. 13.3)

- Choose a number of facilities (storage, depot) to serve a number of customers
- m potential facilities, capacity k_i, $i = 1, \dots, m$
- n customers, demand d_j, $j = 1, \dots, n$
- Fixed cost $f_i > 0$ of opening facility i
- Cost $c_{ij} > 0$ for transporting one unit from facility i to customer j
- $x_{ij} =$ flow from i to j, $\qquad y_i = \begin{cases} 1 & \text{if facility } i \text{ is opened} \\ 0 & \text{otherwise} \end{cases}$

$$
\begin{array}{lrcll}
\min & \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} + \sum_{i=1}^{m} f_i y_i & & & \\
\text{s.t.} & \sum_{j=1}^{n} x_{ij} & \leq & k_i y_i & i = 1, \dots, m \\
 & \sum_{i=1}^{m} x_{ij} & = & d_j & j = 1, \dots, n \\
 & x_{ij} & \geq & 0 & i = 1, \dots, m, \; j = 1, \dots, n \\
 & y_i & \in & \{0, 1\} & i = 1, \dots, m
\end{array}
$$