Chalmers University of Technology
University of Gothenburg
Mathematical Sciences
Optimization
Ann-Brith Strömberg

MVE165
MMG630
Applied Optimization
Assignment information
April 11, 2011

Assignment 2: Maintenance planning

Given below is a mathematical model for finding a maintenance schedule such that the costs of maintaining a system during a limited time period is at minimum. The system consists of several components with economic dependencies. The model together with its problem background are described in the notes of Lecture 10a.

Implementations of the model in AMPL is found on the course homepage:
`www.math.chalmers.se/Math/Grundutb/CTH/mve165/1011/`
Study the AMPL files carefully to get some hints before you start solving the exercises. Call AMPL/CPLEX using the command '`ampl uh.run`' in a Linux command window. The file `uh.run` should be edited in order to solve the different instances of the model, as described in the exercises below.

To pass the assignment you should (in groups of two persons) `(i)` write a report (maximum six pages) that describes and discusses the issues presented in the exercises and questions below. You shall also estimate the number of hours spent on this assignment and note this in your report.

The file containing your report shall be called **Name1-Name2-Ass2.pdf**, where "Name$k$", $k = 1, 2$, is your respective family name.

Do not forget to write the authors' names also inside the report.

The report should be handed in electronically to `anstr@chalmers.se`
at the latest on Monday 2 of May 2011 at 17.00.

You should then `(ii)` write an opposition (maximum 1/2 page) to another group's report which should be handed in
at the latest on Friday 6 of May 2011 at 17.00.

The questions 1–3 below are mandatory. In addition, students aiming at grade 3 or G must answer *at least one* of the questions 4–5, while students aiming at grade 4, 5, or VG must answer *all* the questions.

The mathematical model

Sets and parameters

- $\mathcal{N}$ = the set of components in the system. `(in AMPL: Components)`
- T = the number of time steps in the planning period. `(in AMPL: T)`
- T_i = the life of a new component of type $i \in \mathcal{N}$ (measured in number of time steps). It is assumed that $2 \leq T_i \leq T - 1$. `(in AMPL: U)`
- c_{it} = the cost of a replacement component of type $i \in \mathcal{N}$ at time t (measured in €). For some instances it is assumed that c_{it} is constant over time, i.e., $c_{it} = c_i$, $t = 1, \ldots, T$. `(in AMPL: c)`
- d_t = the cost for a maintenance occasion at time t (measured in €). For some instances it is assumed that d_t is constant over time, i.e., $d_t = d$, $t = 1, \ldots, T$. `(in AMPL: d)`

Decision variables

- $x_{it} = \begin{cases} 1 & \text{if component } i \text{ is replaced at time } t, \\ 0 & \text{otherwise,} \end{cases} \quad i \in \mathcal{N},\ t \in \{1, \ldots, T\}.$
- $z_t = \begin{cases} 1 & \text{if maintenance is made at time } t, \\ 0 & \text{otherwise,} \end{cases} \quad t \in \{1, \ldots, T\}.$

The model

$$\text{minimize} \quad \sum_{t=1}^{T} \left(\sum_{i \in \mathcal{N}} c_{it} x_{it} + d_t z_t \right), \tag{1}$$

$$\text{subject to} \quad \sum_{t=\ell+1}^{T_i+\ell} x_{it} \geq 1, \quad \ell = 0, \ldots, T - T_i,\ i \in \mathcal{N}, \tag{2}$$

$$x_{it} \leq z_t, \quad t = 1, \ldots, T,\ i \in \mathcal{N}, \tag{3}$$

$$x_{it}, z_t \in \{0, 1\},\ t = 1, \ldots, T,\ i \in \mathcal{N}. \tag{4}$$

Description of the model

(1) The objective is to minimize the total cost for the maintenance during the planning period (the time steps $1, \ldots, T - 1$) `(in AMPL: Cost)`.

(2) Each component i must be replaced at least once within each T_i time steps `(in AMPL: ReplaceWithinLife)`.

(3) Components can only be replaced at maintenance occasions `(in AMPL: ReplaceOnlyAtMaintenance)`.

(4) All the variables are required to be binary.

Exercises to perform and questions to answer

1. (a) Solve the model (1)–(4) as implemented in the file `uh-stor.mod` with data from `uh-stor.dat`, letting $T = 125$, and with integer requirements on the variables x_{it} and z_t. *Note that in this instance all costs are time independent, i.e.,* $c_{it} = c_i$ *and* $d_t = d$, $t = 1, \ldots, T$. Relax the integrality on the variables x_{it} and resolve the problem. Then relax the integrality on all variables and resolve the model.

   Compare the solutions obtained and discuss their interpretations. Compare also the computation times (CPU) and explain the differences.

   (b) Solve the model (1)–(4) as implemented in `uh-small.mod` with data from `uh-small.dat`. Relax the integrality constraints on the variables and resolve. Then add the constraint given in `cgcut.mod` and resolve. Compare the solutions obtained and explain their differences.

2. Solve five instances of the model (1)–(4) as implemented in `uh-stor.mod` and `uh-stor.dat` , with the value d_t varying between 0 and 100 (choose these values such that the respective solutions are significantly different). Let $T = 100$.

   (a) Compare the number of maintenance occasions, the number of replaced components, the total cost, and the computation time (in CPU seconds) for these values of d_t. Draw illustrating maintenance schedules for each of the solutions computed.

   (b) Find the smallest value of d_t for which the number of maintenance occasions attains its lowest possible value.

   (c) In practice, one usually wants to replace a component as late as possible within its life. How can the model (1)–(4) be adjusted such that—among all optimal solutions—the one is chosen such that the replacements are made as late as possible?

   (d) Implement these adjustments in the AMPL-files and—for a suitable choice of d_t—compare the schedule with its corresponding schedule from 2a.

3. Solve the model (1)–(4) as implemented in `uh-stor.mod` and `uh-stor.dat`. Let $d = 20$.

   (a) Vary the time horizon between $T = 50$ and $T = 150$ and draw a graph of the computing time (in CPU seconds) as a function of T (use a log-scale). If needed, use the options for limiting the size of the branch–and–bound tree—keeping track of upper and lower bounds on the optimal value (see the file `uh.run`).

   (b) Make an analogous graph for the case when the integrality requirements on the variables are relaxed; vary the time horizon between $T = 50$ and $T = 500$.

   (c) Compare and comment on the complexity properties of the two models solved in 3a and 3b.

(d) Cplex uses the branch–and–bound algorithm, possibly employing presolve steps including heuristics and cutting plane generation. On what does it seem to spend most of the solution time: presolve, finding an optimal (feasible) solution, or verifying its optimality? Discuss your findings in terms of the duality gap for an ILP.

4. Heuristics

   (a) Define a constructive heuristic for the model (1)–(4). Implement in Matlab or AMPL and find a feasible solution to the instance in `uh-stor.mod` and `uh-stor.dat` (letting $d = 20$ and $T = 100$). Apply your algorithm also to the instance of (1)–(4) given by $T = 100$, $|\mathcal{N}| = 4$, $(c_{it})_{i\in\mathcal{N}} = (5, 6, 7, 9)$, $d_t = 10$, $t = 1, \ldots, T$, and $(T_i)_{i\in\mathcal{N}} = (3, 4, 5, 7)$.
   [Hints on how to write AMPL-code in the .run-file is found at `www.ampl.com/NEW/loop1.html` and `www.ampl.com/NEW/loop2.html`.]

   (b) Define a neigbourhood of a feasible solution to the model (1)–(4). A neighbourhood may be defined, e.g., with respect to all the variables or just the variables z_t, $t = 1, \ldots, T$. The search of the neighbourhood involves the solution of subproblems—a well chosen neighbourhood results in subproblems that are "easy" to solve (i.e., in polynomial time). Identify the subproblems resulting from your choice of neighbourhood and describe how these can be efficiently solved.

   (c) Define and implement a local search algorithm for the model (1)–(4) and use it to improve the respective solutions found in 4a.

   (d) Draw illustrating schedules of the solutions found in 4a and 4c and compare with the optimal solution to this instance. Also, draw diagrams showing the objective value as a function of the number of iterations performed in the local search algorithm and CPU seconds, respectively.

5. Assume that it is required that the system (including all of its components) has a remaining life which is at least $r > 0$ time steps at the end of the planning period (i.e., at time $t = T$).

   (a) Add and/or modify constraints to/in the model to accomplish this and solve the resulting model. Start by the model in `uh-stor.mod` with data from `uh-stor.dat` (with $d = 20$ and $T = 100$). Verify that the solution fulfills the requirement stated.

   (b) For five different relevant values of r (these values should be chosen such that the respective solutions become significantly different), compare the total cost for maintenance according to the schedule computed in 5a with that of the "original" (with $r = 0$) one. Comment on the number of maintenance occasions and the number of replaced components and compare with the corresponding numbers from the "original" model (with $r = 0$).

   (c) Which values of r are relevant for this study and why?