MVE165/MMG630, Applied Optimization Lecture 1 Introduction; course map; operations research; modelling optimization applications; graphic solution

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Staff and homepage

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- **Course homepage**
  - `www.math.chalmers.se/Math/Grundutb/CTH/mve165/1011`
  - Details, information on assignmnents and exercises, deadlines, lecture notes, etc
  - Will be updated with new information every course week

Course contents and organization

Contents

- Applications of optimization
- Mathematical modelling
- Theory – properties of models
- Solution techniques – algorithms
- Software solvers

Organization

- Lectures – mathematical optimization theory
- Exercises – use software solvers
- Guest lectures – applications of optimization
- Assignments – modelling, use solvers, written reports, opposition & oral presentation
- Assignment work should be done in groups of two persons

Literature

- Main course book:
  - English version: Optimization (2010)
  - Swedish version: Optimeringslära (2008)

  by J. Lundgren, M. Rönnqvist, and P. Värbrand. Studentlitteratur.
- Exercise book:
  - English version: Optimization Exercises (2010)
  - Swedish version: Optimeringslära Övningsbok (2008)

  by M. Henningsson, J. Lundgren, M. Rönnqvist, and P. Värbrand. Studentlitteratur.
- Cremona/Studentlitteratur/Adlibris/...
- Hand-outs

Examination requirements

- Perform three project assignments in groups of two students
  - For Assignment 3 there are four alternatives.
- Written reports of three assignments
- A written opposition to Assignment 2
- An oral presentation of Assignment 3
- Presence at one oral presentation session
- To be able to receive a grade higher than 3 or G, the written reports and opposition as well as the oral presentation must be of high quality. Students aiming at grade 4, 5, or VG must also pass an oral exam

Overview of the lectures

- Linear optimization, modelling, theory, solution methods, sensitivity analysis
- Discrete optimization models, properties, and solution methods
- Optimization models that can be described as flows in networks, theory, and solution methods
- Multi-objective optimization
- Optimization under uncertainty
- Overview of non-linear optimization models, properties, and solution methods
- Mixes of the above

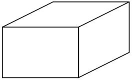
Optimization

"Do something as good as possible"

- **Something:** Which are the decision alternatives?
- **Possible:** What restrictions are there?
- **Good:** What is a relevant optimization criterion?

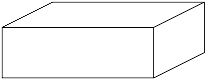
A manufacturing example: Produce tables and chairs from two types of blocks

Small block



$\times 8$

Large block

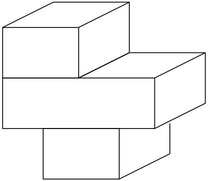


$\times 6$

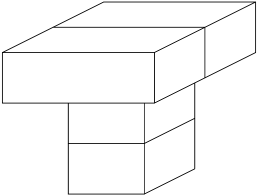
$\Rightarrow$

$\Rightarrow$

Chair



Table



A manufacturing example, continued

- A chair is assembled from one large and two small blocks
- A table is assembled from two blocks of each size
- Only 6 large and 8 small blocks are avaliable
- A table is sold at a revenue of 1600:-
- A chair is sold at a revenue of 1000:-
- Assume that all items produced can be sold and determine an optimal production plan

A mathematical optimization model

Something — What decision alternatives? ⇒ Variables

$$\begin{aligned} x_1 &= \text{number of tables produced and sold} \\ x_2 &= \text{number of chairs produced and sold} \end{aligned}$$

Possible — What restrictions? ⇒ Constraints

$$\begin{array}{rcrcll} 2x_1 & + & x_2 & \leq & 6 & \text{(6 large blocks)} \\ 2x_1 & + & 2x_2 & \leq & 8 & \text{(8 small blocks)} \\ & & x_1,\ x_2 & \geq & 0 & \text{(physical restrictions)} \\ & & (x_1,\ x_2 & \text{integral)} & & \text{(physical restrictions)} \end{array}$$

Good — Relevant optimization criterion? ⇒ Objective function

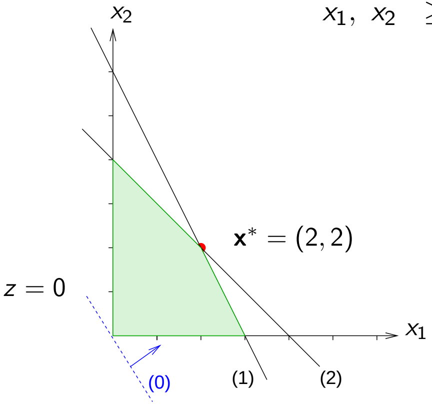
$$\text{maximize} \quad z = 1600x_1 + 1000x_2 \qquad (z = \text{total revenue})$$

Solve the model using LEGO!

- Start at no production: $x_1 = x_2 = 0$
  Use the "best marginal profit" to choose the item to produce
  - x_1 has the highest marginal profit (1600:-/table)
    $\Rightarrow$ produce as many tables as possible
  - At $x_1 = 3$: no more large blocks left
- The marginal value of x_2 is now 200:- since taking apart one table (-1600:-) yields two chairs ($+2000$:-) $\Rightarrow$ 400:-/2 chairs
  - Increase x_2 maximally $\Rightarrow$ decrease x_1
  - At $x_1 = x_2 = 2$: no more small blocks
- The marginal value of x_1 is negative (to build one more table one has to take apart two chairs $\Rightarrow$ -400:-)
  The marginal value of x_2 is -600:- (to build one more chair one table must be taken apart)
  $\Longrightarrow$ Optimal solution: $x_1 = x_2 = 2$

Geometric solution of the model

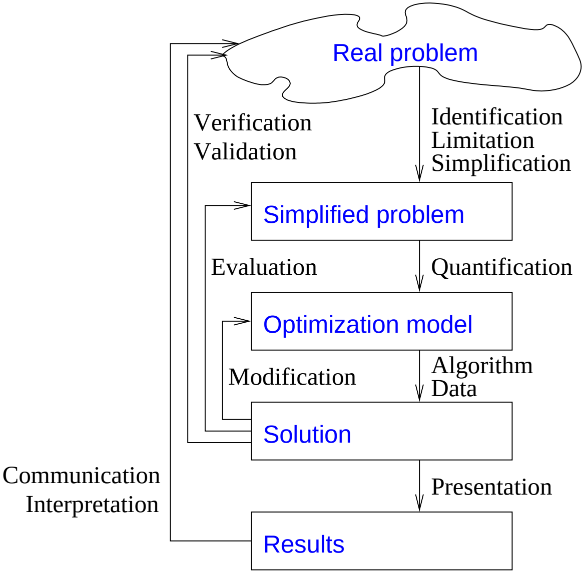
$$\begin{array}{llrcrcl} \text{maximize} & z = & 1600x_1 & + & 1000x_2 & & & (0) \\ \text{subject to} & & 2x_1 & + & x_2 & \leq & 6 & (1) \\ & & 2x_1 & + & 2x_2 & \leq & 8 & (2) \\ & & & & x_1,\ x_2 & \geq & 0 & \end{array}$$



Operations Research (OR) (Swedish: Operationsanalys)

- Scientific view on problem solving regarding complex systems
- *"OR means to prepare a foundation for rational decisions by utilizing systematic scientific methods and — where possible and meaningful — by utilizing quantitative models"*
- The development of OR takes place within two areas:
  - Development of operational research methods, mainly in academic research
  - Application of OR to real problems. Important topics: problem formulation, way of working, method choice
- OR projects are typically performed by a group of persons with different backgrounds
- The problem is considered as a system of components which cooperate and influence each other
- The activities studied are described by models, used to
  - better understand the depicted system,
  - understand the consequences of different decisions, and
  - choose the "best" alternative due to some criterion.

The process of optimization

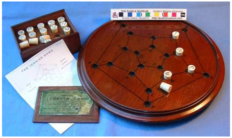


History of Operations Research

- During the second world war the scientific treatment of decision problems became a systematic, established pursuit and an independent concept: Operations Research
- The experience of these operations became very good
- After the war: Operations Research used for civil operations. Initiated fast and broad development of operational research methods and applications; the ideas spread to many countries
- Early operations research include storage planning
- Many problems are addressed with operations research, e.g.
    - Setup telecommunication networks to support service quality
    - Determine bus routes $\Rightarrow$ need as few buses as possible
    - Supply chain management: manage flow of raw materials/products under uncertain demand for end products
    - Freight transportation and delivery systems
    - Scheduling: staff planning, manufacturing, project planning, data traffic (queueing), sports events with television coverage
    - Blending of raw materials in oil refineries

A few moments in optimization history

- Euler (1735): Seven bridges of Köningsberg (graph theory)
- Gauss (1777–1855): 1st optimization technique, steepest descent
- W.R. Hamilton (1857): "icosian game"
  $\Rightarrow$ the travelling salesperson problem
  (Hamilton cycle)



- L.V. Kantorovich (1939): A linear model for optimization of plywood manufactoring and an algorithm for its solution
- George B. Dantzig (1947): Linear programming – the simplex algorithm (exponential time)
  - Program $\Leftrightarrow$ military training and logistics schedules
- Kantorovich and Koopmans (1975): Nobel Memorial Prize in Economic Sciences
- N. Karmarkar (1984) algorithm for linear programming (polynomial time)

Optimization modelling: A production–inventory example

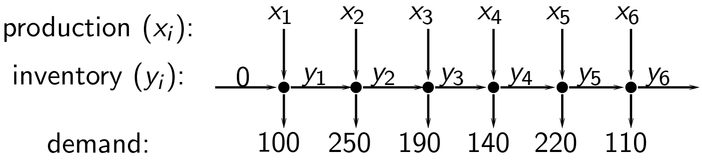
- Commission: Deliver windows over a six-month period
- Demand during the respective months: 100, 250, 190, 140, 220 & 110 units
- Production cost per unit (window): 50 €,45 €, 55 €, 48 €, 52 € & 50 €
- Store a manufactured window from one month to the next at 8 €
- Requirement: Meet the demand and minimize the costs
- Find an optimal production schedule

Define the decision variables

$$x_i = \text{number of units produced in month } i = 1, \ldots, 6$$

$$y_i = \text{units left in the inventory at the end of month } i = 1, \ldots, 6$$

- The "flow" of windows can be illustrated as:



Define the limitations/constraints

- Each month:

initial inventory + production − ending inventory = demand

$$
\begin{array}{ccccccccc}
0 & + & x_1 & - & y_1 & = & 100 & & \\
y_1 & + & x_2 & - & y_2 & = & 250 & & \\
y_2 & + & x_3 & - & y_3 & = & 190 & & \\
y_3 & + & x_4 & - & y_4 & = & 140 & & \\
y_4 & + & x_5 & - & y_5 & = & 220 & & \\
y_5 & + & x_6 & - & y_6 & = & 110 & & \\
 & & x_i & , & y_i & \geq & 0, & i = 1, \ldots, 6 &
\end{array}
$$

Objective function: minimize the costs for production and storage

- Production cost (€):

$$50\ x_1 + 45\ x_2 + 55\ x_3 + 48\ x_4 + 52\ x_5 + 50\ x_6$$

- Inventory cost (€):

$$8\ (y_1 + y_2 + y_3 + y_4 + y_5 + y_6)$$

- Objective:

$$\begin{aligned} \text{minimize} \quad & 50x_1 + 45x_2 + 55x_3 + 48x_4 + 52x_5 + 50x_6 \\ & +8(y_1 + y_2 + y_3 + y_4 + y_5 + y_6) \end{aligned}$$

A complete (general) optimization model

$$\begin{array}{lrcll} \text{minimize} & \sum_{i=1}^{6} c_i x_i + 8 \sum_{i=1}^{6} y_i, & & & \\ \text{subject to} & y_{i-1} + x_i - y_i & = & d_i, & i = 1, \dots, 6, \\ & y_0 & = & 0, & \\ & x_i, y_i & \geq & 0, & i = 1, \dots, 6, \end{array}$$

The vector of demand:

$$d = (d_i)_{i=1}^{6} = (100, 250, 190, 140, 220, 110)$$

The vector of production costs:

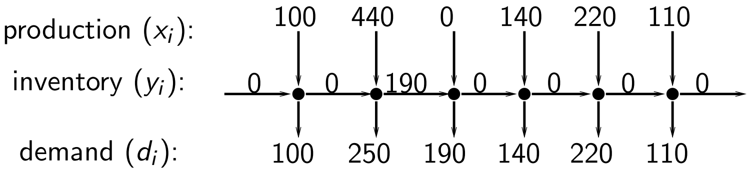
$$c = (c_i)_{i=1}^{6} = (50, 45, 55, 48, 52, 50)$$

.

An optimal solution—optimal production schedule

Optimal production each month:
$x = (x_i)_{i=1}^{6} = (100, 440, 0, 140, 220, 110)$

Optimal inventory each month:
$y = (y_i)_{i=0}^{6} = (0, 0, 190, 0, 0, 0, 0)$



The minimal total cost is 49980 €

Mathematical optimization models

$$\left[\begin{array}{ll} \text{minimize or maximize} & f(x_1,\ldots,x_n) \\ \text{subject to} & g_i(x_1,\ldots,x_n) \left\{\begin{array}{c} \leq \\ = \\ \geq \end{array}\right\} b_i, \quad i=1,\ldots,m \end{array}\right]$$

- $x_1,\ldots,x_n$ are the decision variables
- f and $g_1,\ldots,g_m$ are given functions of the decision variables
- $b_1,\ldots,b_m$ are specified constant parameters
- The functions can be nonlinear, e.g. quadratic, exponential, logarithmic, non-analytic, ...
- In general, linear forms are more tractable than non-linear

Linear optimization models (programs)

- The production inventory model is a linear program (LP), i.e., all relations are described by linear forms
- A general linear program:

$$\left[\begin{array}{lcccl} \text{min or max} & c_1x_1 + \ldots + c_nx_n & & & \\ \text{subject to} & a_{i1}x_1 + \ldots + a_{in}x_n & \left\{\begin{array}{c}\leq\\=\\\geq\end{array}\right\} & b_i, & i = 1, \ldots, m \\ & x_j & \geq & 0, & j = 1, \ldots, n \end{array}\right]$$

- The non-negativity constraints on x_j, $j = 1, \ldots, n$ are not necessary, but usually assumed (reformulation always possible)

Discrete/integer/binary modelling

- A variable is called *discrete* if it can take only a countable set of values, e.g.,
    - Continuous variable: $x \in [0, 8]$ or $0 \leq x \leq 8$
    - Discrete variable: $x \in \{0, 4.4, 5.2, 8.0\}$
    - *Integer* variable: $x \in \{0, 1, 4, 5, 8\}$
- A *binary* variable can only take the values 0 or 1, i.e., all or nothing

  E.g., a wind-mill can produce electricity only if it is built
    - Let $y = 1$ if the mill is built, otherwise $y = 0$
    - Capacity of a mill: C
    - Production $x \leq Cy$ (also limited by wind force etc.)
- In general, models with only continuous variables are more tractable than models with integrality/discrete requirements on the variables, but exceptions exist!

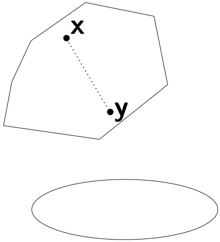
Convex sets

- A set S is convex if, for any elements $\mathbf{x}, \mathbf{y} \in S$ it holds that

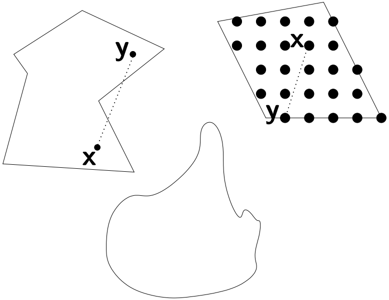
$$\alpha \mathbf{x} + (1 - \alpha)\mathbf{y} \in S \text{ for all } 0 \leq \alpha \leq 1$$

- Examples:

Convex sets



Non-convex sets



$\Rightarrow$ Intersections of linear (in)equalities $\Rightarrow$ convex sets

Convex and concave functions

- A function f is convex on the set S if, for any elements $\mathbf{x}, \mathbf{y} \in S$ it holds that

$$f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \leq \alpha f(\mathbf{x}) + (1-\alpha)f(\mathbf{y}) \text{ for all } 0 \leq \alpha \leq 1$$

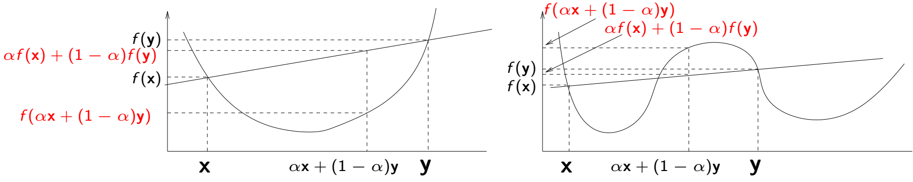
- A function f is concave on the set S if, for any elements $\mathbf{x}, \mathbf{y} \in S$ it holds that

$$f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \geq \alpha f(\mathbf{x}) + (1-\alpha)f(\mathbf{y}) \text{ for all } 0 \leq \alpha \leq 1$$

$\Rightarrow$ Linear functions are convex (and concave)

Convex function

Non-convex function



Global solutions of convex and linear optimization problem

- Let $\mathbf{x}^*$ be a *local* minimizer of a *convex function* over a *convex set*. Then $\mathbf{x}^*$ is also a *global* minimizer.

$\Rightarrow$ Every local optimum of a linear optimization problem is a global optimum

- If a linear optimization problem has any optimal solutions, at least one optimal solution is at an extreme point of the feasible set

$\Rightarrow$ Search for optimal extreme point(s)

- Next lecture: Linear optimization problems and the simplex method