

MVE165/MMG630, Applied Optimization
Lectures 10b and 11a
Minimum cost flow models and algorithms

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Maximum flow models (Ch. 8.6)

- ▶ Consider a district heating network with pipelines that transports energy (in the form of hot water) from a number of sources to a number of destinations
- ▶ The network has several branches and junctions
- ▶ Pipe segment (i, j) has a maximum capacity of K_{ij} units of flow per time unit
- ▶ A pipe can be one- or bidirectional
- ▶ What is the maximum total amount of flow per time unit through this network?
- ▶ Another application of the maximum flow model: evacuation of buildings (also time dynamics)

LP model for maximum flow problems

- ▶ Let x_{ij} denote the amount of flow through pipe segment (i, j) (flow direction $i \rightarrow j$)
- ▶ Let v denote the total flow from the source to the destination
- ▶ Graph: $G = (V, A, K)$ (nodes, directed arcs, arc capacities)
(an undirected edge is here represented by two directed arcs)

$$\begin{aligned} \max \quad & v, \\ \text{s.t.} \quad & \sum_{j:(s,j) \in A} (-x_{sj}) + v = 0, \\ & \sum_{j:(j,t) \in A} x_{jt} - v = 0, \\ & \sum_{i:(i,k) \in A} x_{ik} + \sum_{j:(k,j) \in A} (-x_{kj}) = 0, \quad k \in V \setminus \{s, t\} \\ & x_{ij} \leq K_{ij}, \quad (i, j) \in A \\ & x_{ij} \geq 0, \quad (i, j) \in A \end{aligned}$$

A solution method for maximum flow problems (Edmonds & Karp, 1972)

1. Let $v := 0$ and $x_{ij} := 0$. Arc capacities $u_{ij} := K_{ij}$, $(i, j) \in A$.
2. Find a *maximum capacity* path $P \subset A$ from s to t (modified shortest path algorithm). The capacity of P is $\hat{u} := \min \{ \min \{ u_{ij} \mid (i, j) \in P \}; \min \{ x_{ij} \mid (j, i) \in P \} \}$.
If $\hat{u} = 0$, go to step 4.

3. Update the flows $x_{ij} := \begin{cases} x_{ij} + \hat{u}, & \text{if } (i, j) \in P, \\ x_{ij} - \hat{u}, & \text{if } (j, i) \in P, \\ x_{ij}, & \text{otherwise,} \end{cases}$

the capacities $u_{ij} := \begin{cases} u_{ij} - \hat{u}, & \text{if } (i, j) \in P, \\ u_{ij} + \hat{u}, & \text{if } (j, i) \in P, \\ u_{ij}, & \text{otherwise,} \end{cases}$

and the total flow $v := v + \hat{u}$. Go to step 2.

4. The maximum total flow is v .
The flow solution is given by x_{ij} , $(i, j) \in A$.

LP dual of the maximum flow model

$$\begin{aligned} \text{[Primal]} \quad & \max && v, \\ & \text{s.t.} && \sum_{j:(s,j) \in A} (-x_{sj}) + v = 0, \\ & && \sum_{j:(j,t) \in A} x_{jt} - v = 0, \\ & && \sum_{i:(i,k) \in A} x_{ik} + \sum_{j:(k,j) \in A} (-x_{kj}) = 0, \quad k \in V \setminus \{s, t\} \\ & && 0 \leq x_{ij} \leq K_{ij}, \quad (i, j) \in A \end{aligned}$$

$$\begin{aligned} \text{[Dual]} \quad & \min && \sum_{(i,j) \in A} K_{ij} \gamma_{ij}, \\ & \text{s.t.} && -\pi_i + \pi_j + \gamma_{ij} \geq 0, \quad (i, j) \in A \\ & && \pi_s - \pi_t = 1, \\ & && \pi_k \quad \text{free}, \quad k \in V, \\ & && \gamma_{ij} \geq 0, \quad (i, j) \in A \end{aligned}$$

Maximum flow – Minimum cut theorem

- ▶ An (s, t) -cut is a set of arcs which, when deleted, interrupt all flow in the network between the source s and the sink t
- ▶ The *cut capacity* equals the sum of capacities on all the arcs through the (s, t) -cut
- ▶ Finding the minimum (s, t) -cut is equivalent to solving the dual of the maximum flow problem
- ▶ **Weak duality theorem: Each feasible flow x_{ij} , $(i, j) \in A$, yields a lower bound on v^* . The capacity of each (s, t) -cut yields an upper bound on v^* .**
- ▶ **Strong duality theorem: *value of maximum flow = capacity of minimum cut***

Optimal dual solution – minimum cut

- ▶ Optimal values of the dual variables:

$$\gamma_{ij} = \begin{cases} 1, & \text{if arc } (i,j) \text{ passes through the minimum cut,} \\ 0, & \text{otherwise.} \end{cases}$$

$$\pi_k = \begin{cases} 1, & \text{if node } k \text{ can be reached from } s, \\ 0, & \text{otherwise.} \end{cases}$$

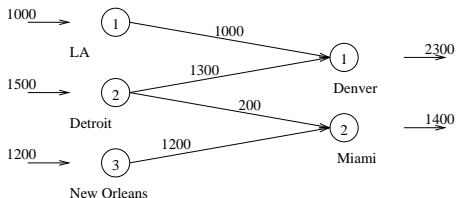
- ▶ How is the minimum cut found using the Edmonds & Karp algorithm?

Transportation models: An example

- ▶ MG Auto has three plants, LA, Detroit, New Orleans, and two distribution centers, Denver and Miami
- ▶ Capacities of the plants: 1000, 1500, and 1200 cars
- ▶ Demands at distributions centers: 2300 and 1400 cars
- ▶ Transportation cost per car between plants and centers:

	Denver	Miami
LA	\$80	\$215
Detroit	\$100	\$108
New Orleans	\$102	\$68

- ▶ Find the cheapest shipping schedule to satisfy the demand



Linear programming formulation of MG Auto

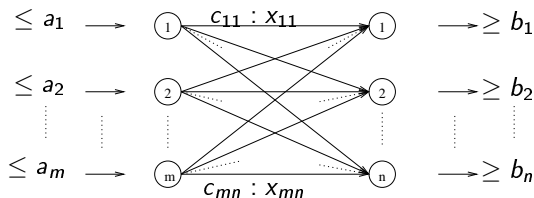
- ▶ Variables: x_{ij} = number of cars sent from plant i to distribution center j

$$\min z = 80x_{11} + 215x_{12} + 100x_{21} + 108x_{22} + 102x_{31} + 68x_{32}$$

$$\begin{array}{rcccccc} \text{s.t.} & x_{11} & +x_{12} & & & & \leq & 1000 & (\text{LA}) \\ & & & x_{21} & +x_{22} & & & \leq & 1500 & (\text{Detr}) \\ & & & & & x_{31} & +x_{32} & \leq & 1200 & (\text{NO}) \\ & x_{11} & & +x_{21} & & +x_{31} & & \geq & 2300 & (\text{Den}) \\ & & x_{12} & & +x_{22} & & +x_{32} & \geq & 1400 & (\text{Mi}) \\ & x_{11}, & x_{12}, & x_{21}, & x_{22}, & x_{31}, & x_{32} & \geq & 0 & \end{array}$$

Definition of the transportation model

- ▶ m sources and n destinations \Leftrightarrow **nodes**
- ▶ a_i = amount of supply at source (node) i , $i = 1, \dots, m$
- ▶ b_j = amount of demand at destination (node) j , $j = 1, \dots, n$
- ▶ **Arc** (i, j) \Leftrightarrow connection from source i to destination j
- ▶ c_{ij} = cost per unit of flow on arc (i, j)
- ▶ **Variables:** x_{ij} = amount of goods shipped on arc (i, j)
- ▶ **Objective:** find $x_{ij} \geq 0$ such that the total cost is minimized while satisfying all supply and demand restrictions



Linear programming transportation model

$$\min z := \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_{j=1}^n x_{ij} \leq a_i, \quad i = 1, \dots, m \quad (\text{supply})$$

$$\sum_{i=1}^m x_{ij} \geq b_j, \quad j = 1, \dots, n \quad (\text{demand})$$

$$x_{ij} \geq 0, \quad i = 1, \dots, m, \quad j = 1, \dots, n$$

- ▶ Feasible solutions exist *if and only if* $\boxed{\sum_i a_i \geq \sum_j b_j}$
- ▶ The constraint matrix has special properties (totally unimodular) \Rightarrow extreme points of the feasible polyhedron are integer (Chapter 8.6.3)

A balanced transportation model

- ▶ What if total amount of demand \neq total amount of supply?
($\sum_i a_i > \sum_j b_j$ (feasible) or $\sum_i a_i < \sum_j b_j$ (infeasible))

$$\begin{array}{ll} \min z := & \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij} \\ \text{s.t.} & \sum_{j=1}^n x_{ij} \leq a_i, \quad i = 1, \dots, m \\ & \sum_{i=1}^m x_{ij} \geq b_j, \quad j = 1, \dots, n \\ & x_{ij} \geq 0, \quad i = 1, \dots, m, j = 1, \dots, n \end{array}$$

⇒ **Balance** the model by dummy source $m+1$ or destination $n+1$

- ▶ Suppose $\sum_i a_i > \sum_j b_j \Rightarrow$ Let $b_{n+1} := \sum_{i=1}^m a_i - \sum_{j=1}^n b_j$

⇒ **Balanced** transportation model—equality constraints

$$\begin{array}{ll} \min z := & \sum_{i=1}^m \sum_{j=1}^{n+1} c_{ij} x_{ij} \\ \text{s.t.} & \sum_{j=1}^{n+1} x_{ij} = a_i, \quad i = 1, \dots, m \\ & \sum_{i=1}^m x_{ij} = b_j, \quad j = 1, \dots, n+1 \\ & x_{ij} \geq 0, \quad i=1, \dots, m, j=1, \dots, n+1 \end{array}$$

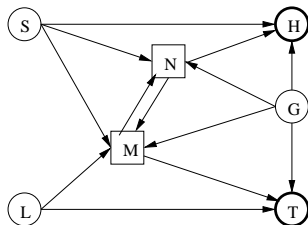
General minimum cost network flow problems

- ▶ A network consist of a set N of *nodes* linked by a set A of *arcs*
- ▶ A distance/cost c_{ij} is associated with each arc
- ▶ Each node i in the network has a net demand d_i
- ▶ Each arc carries an (unknown) amount of flow x_{ij} that is restricted by a maximum capacity $u_{ij} \in [0, \infty]$ and a minimum capacity $\ell_{ij} \in [0, u_{ij}]$
- ▶ The flow through each node must be *balanced*
- ▶ A network flow problem can be formulated as a linear program
- ▶ All extreme points of the feasible set are *integral* – due to the *unimodularity* property of the constraint matrix (see Ch. 8.6.3)

Minimum cost flow in a general network: Example

- ▶ Two paper mills: Holmsund and Tuna
- ▶ Three saw mills: Silje, Graninge and Lunden
- ▶ Two storage terminals: Norrstig and Mellansel

Facility	Supply (m^3)	Demand (m^3)
Silje	2400	
Graninge	1800	
Lunden	1400	
Holmsund		3500
Tuna		2100



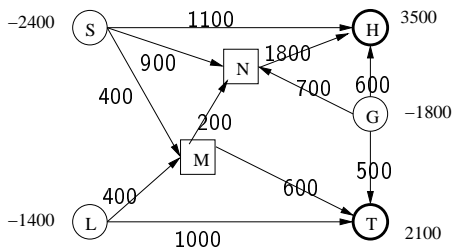
Minimum cost flow in a general network: Example

- ▶ Transportation opportunities:

From	To	Price/m ³	Capacity (m ³)
Silje	Norrstig	20	900
Silje	Mellansel	26	1000
Silje	Holmsund	45	1100
Graninge	Norrstig	8	700
Graninge	Mellansel	14	900
Graninge	Holmsund	37	600
Graninge	Tuna	22	600
Lunden	Mellansel	32	600
Lunden	Tuna	23	1000
Norrstig	Holmsund	11	1800
Norrstig	Mellansel	9	1800
Mellansel	Norrstig	9	1800
Mellansel	Tuna	9	1800

Minimum cost flow in a general network: Example

- ▶ Objective: Minimize transportation costs
- ▶ Satisfy demand
- ▶ Do not exceed the supply
- ▶ Do not exceed the transportation capacities
- ▶ An optimal solution



Minimum cost flow in a general network: Example

$$\begin{aligned}
 \min z := & 20x_{SN} + 26x_{SM} + 45x_{SH} + 8x_{GN} + 14x_{GM} \\
 & + 37x_{GH} + 22x_{GT} + 32x_{LM} + 23x_{LT} + 11x_{NH} \\
 & + 9x_{NM} + 9x_{MN} + 9x_{MT} \\
 \text{subject to} & \quad -x_{SN} - x_{SM} - x_{SH} = -2400 \quad (\text{Silje}) \\
 & \quad -x_{GN} - x_{GM} - x_{GH} - x_{GT} = -1800 \quad (\text{Graninge}) \\
 & \quad \quad -x_{LM} - x_{LT} = -1400 \quad (\text{Lunden}) \\
 & \quad \quad x_{SN} + x_{GN} + x_{MN} - x_{NM} - x_{NH} = 0 \quad (\text{Norrstig}) \\
 & \quad x_{SM} + x_{LM} + x_{GM} + x_{NM} - x_{MN} - x_{MT} = 0 \quad (\text{Mellansel}) \\
 & \quad \quad x_{SH} + x_{GH} + x_{NH} = 3500 \quad (\text{Holmsund}) \\
 & \quad \quad x_{GT} + x_{LT} + x_{MT} = 2100 \quad (\text{Tuna}) \\
 & \quad 0 \leq x_{SN} \leq 900 \\
 & \quad 0 \leq x_{SM} \leq 1000 \\
 & \quad 0 \leq x_{SH} \leq 1100 \\
 & \quad 0 \leq x_{GN} \leq 700 \\
 & \quad 0 \leq x_{GM} \leq 900 \\
 & \quad 0 \leq x_{GH} \leq 600 \\
 & \quad 0 \leq x_{GT} \leq 600 \\
 & \quad 0 \leq x_{LM} \leq 600 \\
 & \quad 0 \leq x_{LT} \leq 1000 \\
 & \quad 0 \leq x_{NH} \leq 1800 \\
 & \quad 0 \leq x_{NM} \leq 1800 \\
 & \quad 0 \leq x_{MN} \leq 1800 \\
 & \quad 0 \leq x_{MT} \leq 1800
 \end{aligned}$$

- ▶ The columns \mathbf{A}_j of the equality constraint matrix ($\mathbf{Ax} = \mathbf{b}$) have one 1-element, one -1-element; the remaining elements are 0

Minimum cost flows in general networks: LP model

- ▶ $G = (N, A)$ is a network with nodes N and arcs A , $|N| = n$
- ▶ x_{ij} is the amount of flow on the arc from node i to node j ,
- ▶ ℓ_{ij} and u_{ij} are lower and upper limits for the flow on arc (i, j) ,
- ▶ c_{ij} is the cost per unit of flow on arc (i, j) , and
- ▶ d_i is the demand in node i

$$\begin{aligned} \min \quad & \sum_{(i,j) \in A} c_{ij} x_{ij}, \\ \text{s.t.} \quad & \sum_{i:(i,k) \in A} x_{ik} - \sum_{j:(k,j) \in A} x_{kj} = d_k, \quad k \in N, \\ & \ell_{ij} \leq x_{ij} \leq u_{ij}, \quad (i,j) \in A. \end{aligned}$$

Minimum cost flows in general networks: LP model and dual

The linear optimization model:

$$\begin{aligned} \min \quad & \sum_{(i,j) \in A} c_{ij} x_{ij}, \\ \text{s.t.} \quad & \sum_{i:(i,k) \in A} x_{ik} - \sum_{j:(k,j) \in A} x_{kj} = d_k, \quad k \in N, \\ & \ell_{ij} \leq x_{ij} \leq u_{ij}, \quad (i,j) \in A. \end{aligned}$$

Linear programming dual:

$$\begin{aligned} \max \quad & \sum_{k \in N} d_k y_k + \sum_{(i,j) \in A} (\ell_{ij} \alpha_{ij} - u_{ij} \beta_{ij}), \\ \text{s.t.} \quad & y_j - y_i + \alpha_{ij} - \beta_{ij} = c_{ij}, \quad (i,j) \in A, \\ & \alpha_{ij}, \beta_{ij} \geq 0, \quad (i,j) \in A. \end{aligned}$$

The simplex method for minimum cost network flows (Ch. 8.7)

- ▶ A solution is optimal if
 - ▶ the primal and dual solutions are feasible and
 - ▶ the complementary conditions are fulfilled
- ▶ Reduced cost: $\bar{c}_{ij} = c_{ij} + y_i - y_j$
- ▶ Complementary conditions, $(i, j) \in A$
 - ▶ $\alpha_{ij}(x_{ij} - \ell_{ij}) = 0$
 - ▶ $\beta_{ij}(u_{ij} - x_{ij}) = 0$
 - ▶ $x_{ij}(\bar{c}_{ij} - \alpha_{ij} + \beta_{ij}) = 0$
- ▶ Assume that $\ell_{ij} < u_{ij}$.
- ▶ A feasible solution x_{ij} , $(i, j) \in A$, is optimal if the following hold:
 - ▶ $x_{ij} = u_{ij} \Rightarrow \alpha_{ij} = 0 \Rightarrow$ Reduced cost: $\bar{c}_{ij} = -\beta_{ij} \leq 0$
 - ▶ $x_{ij} = \ell_{ij} \Rightarrow \beta_{ij} = 0 \Rightarrow$ Reduced cost: $\bar{c}_{ij} = \alpha_{ij} \geq 0$
 - ▶ $\ell_{ij} < x_{ij} < u_{ij} \Rightarrow \alpha_{ij} = \beta_{ij} = 0 \Rightarrow$ Reduced cost: $\bar{c}_{ij} = 0$

The simplex method for minimum cost network flows

- ▶ The arc (i, j) corresponds to the variable x_{ij} , $(i, j) \in A$
- ▶ A *basic solution* is characterized by the following;
 - ▶ If $\ell_{ij} < x_{ij} < u_{ij} \Rightarrow$ the arc (i, j) is in the *basis*
 $\Leftrightarrow x_{ij}$ is a basic variable
 - ▶ If $x_{ij} = \ell_{ij}$ or $x_{ij} = u_{ij} \Rightarrow$ the arc (i, j) *may* be in the *basis*
 $\Leftrightarrow x_{ij}$ *may be* a basic variable
 - ▶ There are exactly $n - 1$ basic arcs which form a *spanning tree* in G (one primal equation is a linear combination of the rest and can thus be removed)

The simplex method for minimum cost flows

1. Find a feasible solution (a spanning tree of basic arcs)
2. Compute reduced costs $\bar{c}_{ij} = c_{ij} + y_i - y_j$ for all non-basic arcs
3. Check termination criteria: If, for every arc (i, j) ,
 - ▶ either: $\bar{c}_{ij} = 0$ and $\ell_{ij} \leq x_{ij} \leq u_{ij}$,
 - ▶ or: $\bar{c}_{ij} < 0$ and $x_{ij} = u_{ij}$,
 - ▶ or: $\bar{c}_{ij} > 0$ and $x_{ij} = \ell_{ij}$

hold, then STOP. x_{ij} , $(i, j) \in A$ is an optimal solution

4. *Entering variable (arc)*: $(p, q) \in \arg \max_{(i,j) \in I} |\bar{c}_{ij}|$
 $I =$ the set of non-basic arcs *not* fulfilling the conditions in 3.
5. *Leaving variable (arc)*: Send flow along the cycle defined by the current *basis* (spanning tree) and the arc (p, q) . The arc (i, j) whose flow x_{ij} first reaches u_{ij} or ℓ_{ij} leaves the basis.
6. Go to step 2

The assignment model (Ch. 13.5)

- ▶ A special case of the network flow model (and of the transportation model)
- ▶ Given n persons and n jobs
- ▶ Given further the cost c_{ij} of assigning person i to job j
- ▶ Binary variables $x_{ij} = 1$ if person i does job j and $x_{ij} = 0$ otherwise
- ▶ Find the cheapest assignment of persons to jobs such that all jobs are done

$$\begin{array}{ll} \min & \sum_{ij} c_{ij}x_{ij} \\ \text{s.t.} & \sum_j x_{ij} = 1 \quad \forall i \\ & \sum_i x_{ij} = 1 \quad \forall j \\ & x_{ij} \geq 0 \quad \forall i, j \end{array}$$

- ▶ The optimal solution is binary (due to the totally unimodular constraint matrix)

An assignment example

- ▶ 3 children: John, Karin and Tina
- ▶ 3 tasks: mow, paint and wash.
- ▶ Given further a “cost” (time, uncomfot,...) for each combination of child/task
- ▶ How should the parents distribute the tasks to minimize the cost?

	Mow	Paint	Wash
John	15	10	9
Karin	9	15	10
Tina	10	12	8

- ▶ Choose exactly one element in each row and one in each column