MVE165/MMG630, Applied Optimization
Lecture 13a
Optimization under uncertainty

Ann-Brith Strömberg

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Applied optimization — optimization under uncertainty

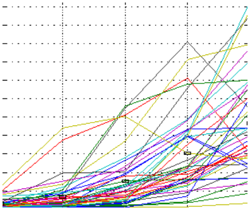
- Decisions may have to be made before information is known
- **Examples**
  - Investments — uncertain future profits (Ass 3d)
  - Hydro and wind power production — uncertain weather conditions (Ass 3b)
  - Maintenance planning — uncertain lives of components (Ass 2)
  - Energy systems —uncertain climate policy/supply/demand/... (Ass 1)
- Represent uncertain data by (discrete) probability distributions
- Consider decisions to make after the information is revealed (*stochastic programming*)
- **Literature on optimization under uncertainty**
  Copies from the book *Stochastic Programming* by P. Kall and S.W. Wallace (second edition, 1994), pp. 1–15, handed out

Deterministic vs. stochastic optimization models

- In a *deterministic* optimization model, *uncertain* parameters are represented by, e.g., (empirical) averages
- The weakness is that the *prediction is considered as a truth* and desicions are made as if the future was completely known
- Optimization tends to *augment errors in the data* when uncertain data is replaced by predictions — *"Do something as good as possible"*
- We need a methodology for handling optimization under uncertainty

Scenario approaches vs. optimization under uncertainty

- **Common approach:** Handle deficiency of deterministic optimization models by identifying *scenarios* representing the uncertainty—solve one deterministic model for each scenario



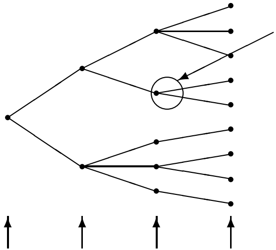
⇒ May yield some information about the variations of the solution,

- But each decision proposal presumes *perfect information* about the future

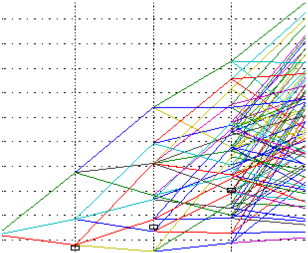
- **Optimization under uncertainty (***stochastic programming***)**
- Uncertain parameters represented by *stochastic variables*
- Discretization necessary for the solvability of the models ⇒ discrete (approximate) probability distributions

A decision must be made prior all data being known

- Several time stages $\Rightarrow$ a scenario *tree*



A scenario tree with three realizations each time step:



- Stochastic variables are realized *between* decision time points.
- Goal: optimize the *expected value over the scenario tree* of, e.g., the revenue

Deterministic optimization vs. optimization under uncertainty

Deterministic optimization

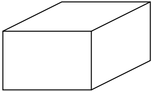
- All parameters and conditions are assumed known for sure

Optimization under uncertainty

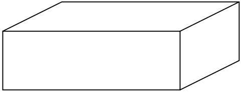
- Decisions are based on observations (previous outcomes) and under *uncertainty of future outcomes*
- A decision *must not depend* on outcomes not yet revealed
- A node in the tree $\Longleftrightarrow$ a vector of decision variables
- In a specific node, the remaining future uncertainty is represented by the sub-tree rooted in that node
- Typically, only the decision associated to the root node is implemented—next time stage a new model is solved—a so called *rolling horizon*
- The magnitude of an optimization problem *increases* when uncertainties are modeled explicitly

The LEGO furniture factory revisited

Small block, purchase price: 100



Large block, purchase price: 200



Chair: [demand: $50 + \xi_2$; sales price: $(5 + \eta_2) \cdot 100$; prod. cost: 50]

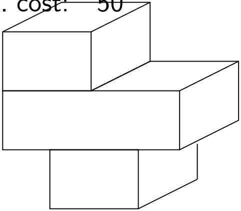
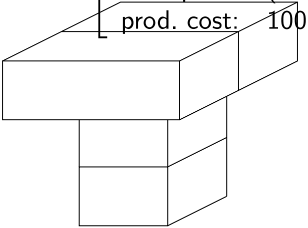


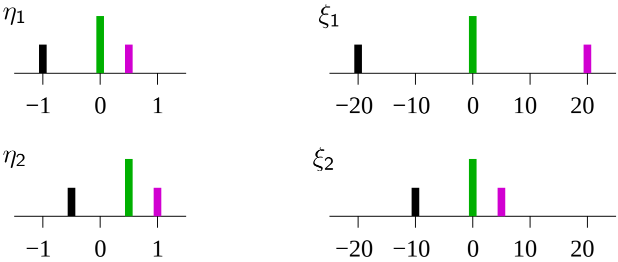
Table: [demand: $100 + \xi_1$; sales price: $(8 + \eta_1) \cdot 100$; prod. cost: 100]



The *stochastic parameters* η_1, η_2, ξ_1, and ξ_2 are assumed to have discrete probability functions

Discrete probability functions

- The values $\eta_1 \in \{-1, 0, 0.5\}$, $\eta_2 \in \{-0.5, 0.5, 1\}$, $\xi_1 \in \{-20, 0, 20\}$, $\xi_2 \in \{-10, 0, 5\}$ are achieved with probabilities $\{\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\}$



- **Assumption here:** Low/medium/high demand levels correspond to low/medium/high price levels

$\Rightarrow$ The four stochastic parameters are dependent

$\Rightarrow$ Totally three scenarios

Definition of variables

- The demand and the selling prices are *not known* when the blocks are to be purchased
- The purchase budget is 80000
- Variables:

$$
\begin{aligned}
x_1 &= \text{\# of large blocks purchased} \\
x_2 &= \text{\# of small blocks purchased} \\
y_1 &= \text{\# tables produced} \\
y_2 &= \text{\# chairs produced} \\
v_1 &= \text{\# tables sold} \\
v_2 &= \text{\# chairs sold}
\end{aligned}
$$

- The values of the purchase variables x_1 and x_2 must be *decided on before* the demand and selling prices are known
- Production (y_1 and y_2) and sales (v_1 and v_2) are decided on later

Mathematical model

Minimize *purchase cost* plus *production cost* minus *sales revenue*

$$\text{minimize}_{x,y,v} \quad z := 100 \cdot [2x_1 + x_2 + y_1 - (8+\eta_1)v_1 + 0.5y_2 - (5+\eta_2)v_2]$$

$$\begin{array}{lrrrrll}
\text{subject to} & 2x_1 & +x_2 & & & \leq 800 & \text{(budget)} \\
& x_1 & & -2y_1 & -y_2 & \geq 0 & \text{(large blocks used} \leq \text{purchased)} \\
& & x_2 & -2y_1 & -2y_2 & \geq 0 & \text{(small blocks used} \leq \text{purchased)} \\
& & & y_1 - v_1 & & \geq 0 & \text{(tables sold} \leq \text{produced)} \\
& & & & y_2 - v_2 & \geq 0 & \text{(chairs sold} \leq \text{produced)} \\
& & & v_1 & & \leq 100 + \xi_1 & \text{(tables sold} \leq \text{demand)} \\
& & & & v_2 & \leq 50 + \xi_2 & \text{(chairs sold} \leq \text{demand)} \\
& x_1, & x_2, & y_1, v_1 & y_2, v_2 & \geq 0 & \text{(integer)}
\end{array}$$

Deterministic (expected value) solution

- Assume that the stochastic parameters attain their respective expected values:
  - $E(\eta_1) = \frac{1}{4} \cdot (-1) + \frac{1}{2} \cdot 0 + \frac{1}{4} \cdot 0.5 = -0.125$
  - $E(\eta_2) = \frac{1}{4} \cdot (-0.5) + \frac{1}{2} \cdot 0.5 + \frac{1}{4} \cdot 1 = 0.375$
  - $E(\xi_1) = \frac{1}{4} \cdot (-20) + \frac{1}{2} \cdot 0 + \frac{1}{4} \cdot 20 = 0$
  - $E(\xi_2) = \frac{1}{4} \cdot (-10) + \frac{1}{2} \cdot 0 + \frac{1}{4} \cdot 5 = -1.25$
- Replace the stochastic parameters in the mathematical model by their expeced values.

The expected value solution

$$
\begin{array}{llllllll}
\text{minimize}_{x,y,z} & z := 100 \cdot [2x_1 + x_2 + y_1 - 7.875v_1 + 0.5y_2 - 5.375v_2] \\
\text{subject to} & 2x_1 & +x_2 & & & \leq 800 \\
& x_1 & & -2y_1 & -y_2 & \geq 0 \\
& & x_2 & -2y_1 & -2y_2 & \geq 0 \\
& & & y_1 - v_1 & & \geq 0 \\
& & & & y_2 - v_2 & \geq 0 \\
& & & v_1 & & \leq 100 \\
& & & & v_2 & \leq 48.75 \\
& x_1, & x_2, & y_1, v_1, & y_2, v_2 & \geq 0 & \text{(integer)}
\end{array}
$$

- Solution: $x_1 = 248.75$, $x_2 = 297.5$, $y_1 = v_1 = 100$, $y_2 = v_2 = 48.75$, $z := -13016$ (minus the profit)
- **Deterministic solution:**
  - Purchase ≈ 249 large and ≈ 298 small blocks
  - Produce and sell ≈ 100 tables and ≈ 49 chairs
  - Profit: 13016

OBS: Infeasible at the lowest demand scenario (80 tables, 40 chairs)!

A hedging deterministic optimization model

- Choose $\eta_1 = -1$, $\eta_2 = -0.5$, $\xi_1 = -20$, and $\xi_2 = -10$

$$
\begin{array}{lrrrrl}
\text{minimize}_{x,y,z} & \multicolumn{5}{l}{z := 100 \cdot [2x_1 + x_2 + y_1 - 7v_1 + 0.5y_2 - 4.5v_2]} \\
\text{subject to} & 2x_1 & +x_2 & & & \leq 800 \\
 & x_1 & & -2y_1 & -y_2 & \geq 0 \\
 & & x_2 & -2y_1 & -2y_2 & \geq 0 \\
 & & & y_1 - v_1 & & \geq 0 \\
 & & & & y_2 - v_2 & \geq 0 \\
 & & & v_1 & & \leq 80 \\
 & & & & v_2 & \leq 40 \\
 & x_1, & x_2, & y_1, v_1, & y_2, v_2 & \geq 0 \quad \text{(integer)}
\end{array}
$$

- Solution: $x_1 = 152.1$, $x_2 = 181.8$, $y_1 = v_1 = 61.22$, $y_2 = v_2 = 29.67$, $z = 0$
- **Deterministic solution:**
  - Purchase $\approx$ 152 large and $\approx$ 182 small blocks
  - Produce and sell $\approx$ 61 tables and $\approx$ 30 chairs
  - Profit: 0

A stochastic optimization model

Stage 1: The purchase decision takes the possible outcomes of demand and selling prices into consideration, with their respective probabilities, and the corresponding decisions on production/sales to make later on

- Three different *scenarios*: low/medium/high level of prices and demand: $\eta^1 = [-1, -0.5]$, $\xi^1 = [-20, -10]$, $\eta^2 = [0, 0.5]$, $\xi^2 = [0, 0]$, $\eta^3 = [0.5, 1]$, $\xi^3 = [20, 5]$

Stage 2: When the decisions on production/sales are to be made, the levels of prices and demand will be revealed

- Also the decided purchase of raw material is known

$\Rightarrow$ Optimize with respect to the outcome of the stochastic parameters and the decisions from stage 1 (*recourse*)

The first stage decision

- Minimize the purchase cost minus the *expected future profit*
- Decide on how many blocks to purchase (x)
- Consider the possible future outcomes of the demand (ξ) and price (η) levels and the decisions on the production ($y(x, \xi, \eta)$) and sales ($v(x, \xi, \eta)$)

$$
\begin{array}{lrll}
\text{minimize}_x & z := 100 \cdot [2x_1 + x_2 & - E_{\xi,\eta} Q(x, \xi, \eta)] & \text{(convex in } x\text{)} \\
\text{subject to} & 2x_1 + x_2 & \leq 800 & \text{(purchase} \leq \text{budget)} \\
 & x_1, x_2 & \geq 0 \quad \text{(integer)} &
\end{array}
$$

- $E_{\xi,\eta} Q(x, \xi, \eta)$ denotes the *expected value of the future profit*, which is computed in stage 2

The second stage decisions

- Maximize future profit (sales revenues minus production costs)
- Decide on production and sales for each outcome of the price (η) and demand (ξ) and for each purchase decision (x)

$$
Q(x,\xi,\eta) = \left(\begin{array}{rrrll}
\text{maximize}_{y,v} & -y_1 + (8+\eta_1)v_1 - 0.5y_2 + (5+\eta_2)v_2 & & & \\
\text{subject to} & 2y_1 & +y_2 & \leq x_1 & \\
& 2y_1 & +2y_2 & \leq x_2 & \\
& v_1 & & \leq y_1 & \\
& & v_2 & \leq y_2 & \\
& v_1 & & \leq 100+\xi_1 & \\
& & v_2 & \leq 50+\xi_2 & \\
& y_1, v_1 & y_2, v_2 & \geq 0 & \text{(integer)}
\end{array} \right)
$$

Expected future profits—the second stage decisions

$$E_{\xi,\eta} Q(x,\xi,\eta) = \frac{1}{4} Q(x,\xi^1,\eta^1) + \frac{1}{2} Q(x,\xi^2,\eta^2) + \frac{1}{4} Q(x,\xi^3,\eta^3)$$

$$= \frac{1}{4}\begin{pmatrix} \max -y_1^1 + 7v_1^1 - 0.5y_2^1 + 4.5v_2^1 \\ \text{subject to} \quad 2y_1^1 + y_2^1 \le x_1 \\ 2y_1^1 + 2y_2^1 \le x_2 \\ v_1^1 \le \min\{y_1^1, 80\} \\ v_2^1 \le \min\{y_2^1, 40\} \\ v_1^1, v_2^1, y_1^1, y_2^1 \ge 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \max -y_1^2 + 8v_1^2 - 0.5y_2^2 + 5.5v_2^2 \\ \text{subject to} \quad 2y_1^2 + y_2^2 \le x_1 \\ 2y_1^2 + 2y_2^2 \le x_2 \\ v_1^2 \le \min\{y_1^2, 100\} \\ v_2^2 \le \min\{y_2^2, 50\} \\ v_1^2, v_2^2, y_1^2, y_2^2 \ge 0 \end{pmatrix}$$

$$+ \frac{1}{4}\begin{pmatrix} \max -y_1^3 + 8.5v_1^3 - 0.5y_2^3 + 6v_2^3 \\ \text{subject to} \quad 2y_1^3 + y_2^3 \le x_1 \\ 2y_1^3 + 2y_2^3 \le x_2 \\ v_1^3 \le \min\{y_1^3, 120\} \\ v_2^3 \le \min\{y_2^3, 55\} \\ v_1^3, v_2^3, y_1^3, y_2^3 \ge 0 \end{pmatrix}$$

A deterministic equivalent model

$$\begin{aligned}
\text{minimize} \quad z := {} & 2x_1 + x_2 + \tfrac{1}{4}\left(y_1^1 - 7v_1^1 + 0.5y_2^1 - 4.5v_2^1\right) \\
& + \tfrac{1}{2}\left(y_1^2 - 8v_1^2 + 0.5y_2^2 - 5.5v_2^2\right) + \tfrac{1}{4}\left(y_1^3 - 8.5v_1^3 + 0.5y_2^3 - 6v_2^3\right)
\end{aligned}$$

$$\begin{array}{llllllllllll}
\text{subject to} & 2x_1 & +x_2 & & & & & & & & & \leq 800 \\
& -x_1 & & +2y_1^1 & +y_2^1 & & & & & & & \leq 0 \\
& & -x_2 & +2y_1^1 & +2y_2^1 & & & & & & & \leq 0 \\
& & & v_1^1 & & & & & & & & \leq \min\{y_1^1, 80\} \\
& & & & v_2^1 & & & & & & & \leq \min\{y_2^1, 40\} \\
& -x_1 & & & & +2y_1^2 & +y_2^2 & & & & & \leq 0 \\
& & -x_2 & & & +2y_1^2 & +2y_2^2 & & & & & \leq 0 \\
& & & & & v_1^2 & & & & & & \leq \min\{y_1^2, 100\} \\
& & & & & & v_2^2 & & & & & \leq \min\{y_2^2, 50\} \\
& -x_1 & & & & & & +2y_1^3 & +y_2^3 & & & \leq 0 \\
& & -x_2 & & & & & +2y_1^3 & +2y_2^3 & & & \leq 0 \\
& & & & & & & v_1^3 & & & & \leq \min\{y_1^3, 120\} \\
& & & & & & & & v_2^3 & & & \leq \min\{y_2^3, 55\} \\
& x_1, & x_2, & y_1^1, y_2^1, & v_1^1, v_2^1, & y_1^2, y_2^2, & v_1^2, v_2^2, & y_1^3, y_2^3 & v_1^3, v_2^3 & & & \geq 0 \ \text{(integer)}
\end{array}$$

The magnitude increases considerably with # scenarios!

Solution to the optimization model that takes the uncertainty into consideration

- First stage solution: $x = (200, 250)$

$\Rightarrow$ Objective value (minus expected profit): $z = -10687$

- Second stage solution: $y^1 = v^1 = (80, 40)$,
  $y^2 = v^2 = y^3 = v^3 = (75, 50)$
  - If scenario 1 occurs (low) the profit becomes:
    $-100 \cdot (2 \cdot 200 + 250 + 80 - 7 \cdot 80 + 0.5 \cdot 40 - 4.5 \cdot 40) = -1000$
  - If scenario 2 occurs (medium) the profit becomes:
    $-100 \cdot (2 \cdot 200 + 250 + 75 - 8 \cdot 75 + 0.5 \cdot 50 - 5.5 \cdot 50) = 12500$
  - If scenario 3 occurs (high) the profit becomes:
    $-100 \cdot (2 \cdot 200 + 250 + 75 - 8.5 \cdot 75 + 0.5 \cdot 50 - 6 \cdot 50) = 18750$
  - Expected profit: $\frac{-1000}{4} + \frac{12500}{2} + \frac{18750}{4} = 10687$

What if we would solve the deterministic model (expected value solution) for the first stage decision (x) and adjust the second stage solution ((y, v)) to the actual scenario observed in the second stage?

1. Solve the deterministic model $\Rightarrow \overline{x}_1 = 248.75$, $\overline{x}_2 = 297.5$
2. Compute the future profit $Q(\overline{x}, \xi, \eta)$ for each scenario
3. The expected value of the expected value (deterministic) solution is: $z = 100 \cdot [2\overline{x}_1 + \overline{x}_2 - E_{\xi,\eta} Q(\overline{x}, \xi, \eta)]$ (next page)
4. Second stage solutions (three different scenarios):
   $y^1 = v^1 = (80, 40)$ $y^2 = v^2 = y^3 = v^3 = (100, 48.75)$
   $\Rightarrow$ The expected profit of the expected value solution: 9140
5. The value of the stochastic solution: $10687 - 9140 = 1547 > 0$

The expected profit increases when taking the uncertainties under consideration already in the formulation of the model

Expected future profit for $\overline{x} = (248.75, 297.5)$

$$E_{\xi,\eta} Q(\overline{x}, \xi, \eta) = \frac{1}{4} Q(\overline{x}, \xi^1, \eta^1) + \frac{1}{2} Q(\overline{x}, \xi^2, \eta^2) + \frac{1}{4} Q(\overline{x}, \xi^3, \eta^3)$$

$$= \frac{1}{4} \begin{pmatrix} \max -y_1^1 + 7v_1^1 - 0.5y_2^1 + 4.5v_2^1 \\ \text{subject to} \quad 2y_1^1 + y_2^1 \leq 248.75 \\ 2y_1^1 + 2y_2^1 \leq 297.5 \\ v_1^1 \leq \min\{y_1^1, 80\} \\ v_2^1 \leq \min\{y_2^1, 40\} \\ v_1^1, v_2^1, y_1^1, y_2^1 \geq 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \max -y_1^2 + 8v_1^2 - 0.5y_2^2 + 5.5v_2^2 \\ \text{subject to} \quad 2y_1^2 + y_2^2 \leq 248.75 \\ 2y_1^2 + 2y_2^2 \leq 297.5 \\ v_1^2 \leq \min\{y_1^2, 100\} \\ v_2^2 \leq \min\{y_2^2, 50\} \\ v_1^2, v_2^2, y_1^2, y_2^2 \geq 0 \end{pmatrix}$$

$$+ \frac{1}{4} \begin{pmatrix} \max -y_1^3 + 8.5v_1^3 - 0.5y_2^3 + 6v_2^3 \\ \text{subject to} \quad 2y_1^3 + y_2^3 \leq 248.75 \\ 2y_1^3 + 2y_2^3 \leq 297.5 \\ v_1^3 \leq \min\{y_1^3, 120\} \\ v_2^3 \leq \min\{y_2^3, 55\} \\ v_1^3, v_2^3, y_1^3, y_2^3 \geq 0 \end{pmatrix}$$