Assignment 3a: The Traveling Salesman Problem

MVE165/MMG630: Applied Optimization 2011

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Problem background

Mathematical formulation

Algorithms
- Heuristics
- Relaxation algorithms

Assignment

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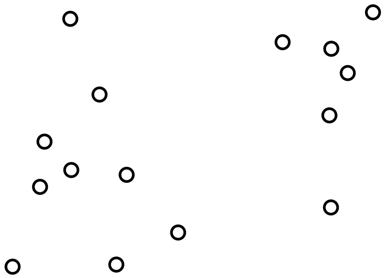
- Stated as a mathematical problem in the 1930's:

Given a list of cities and their pairwise distances, find the shortest possible tour that visits each city exactly once.

Example



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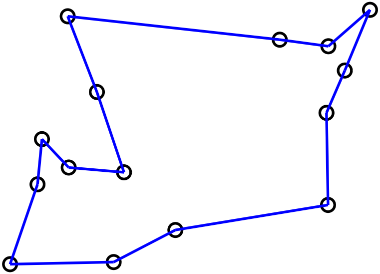

Applications

The TSP has many applications.

- Logistics.
- Production (microchips).
- DNA-sequencing.
- Agriculture.
- Internet planning.

Complexity

The TSP is a *NP-complete* problem (the decision version of it).

- No polynomial algorithm for solving it to optimality.
- Exponential in the number of cities.
- $(N-1)!$ different tours.

Large problems solved to optimality

- VLSI problem (85,900 nodes). Solved 2004, first studied 1991.
- Shortest tour between all the cities in Sweden (24,978 cities) found in 2001. Length $\approx$ 72,500 km

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- Metric TSP (triangle inequality satisfied)

$$c_{ik} + c_{kj} \geq c_{ij}, \quad \forall \text{ cities } i, j, k$$

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- Introduce binary variables x_{ij} where

$$x_{ij} = \begin{cases} 1 & \text{if there is a connection between city } i \text{ and city } j \\ 0 & \text{otherwise} \end{cases}$$

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minimize $\displaystyle\sum_{(i,j)\in\mathcal{L}} c_{ij}x_{ij},$

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- From each subset of cities, we must travel at least once to another city not included in the subset.

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One problem with the formulation is the number of subtour constraints in both formulations are $\approx 2^N$.

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Together these algorithms can provide us with an *optimality interval*

Construcive heuristics

Strategies, rules for obtaining feasible solutions.

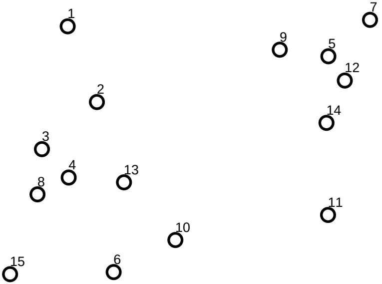
Deterministic

- Based on simple rules for choosing tours: *Nearest neighbour*, *Insertion heuristics*
- Based on solving easier subproblems. *MST-heuristic*, *Christophides heuristic*

Probabilistic

- Based on stochastic rules for choosing tours.
- *Genetic algorithms*, *simulated annealing*, *ant colony optimization*.

Nearest neighbour heuristic



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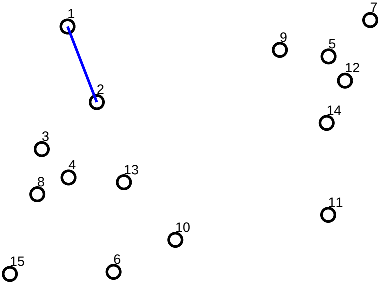

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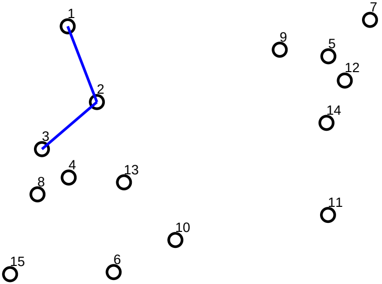

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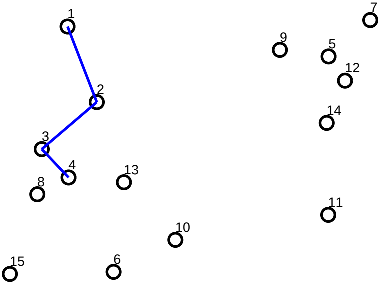

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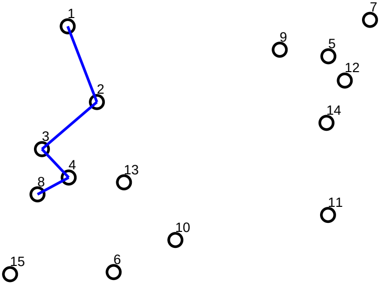

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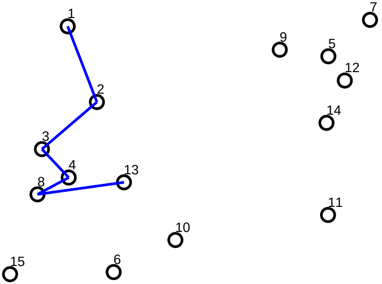

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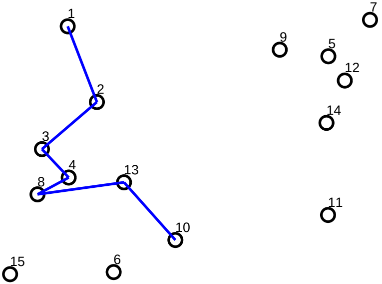

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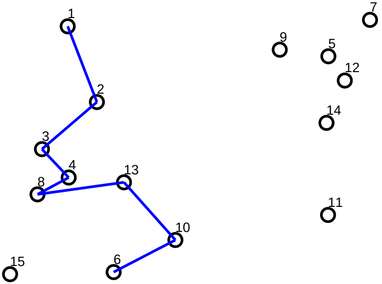

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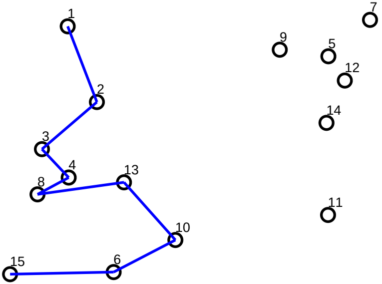

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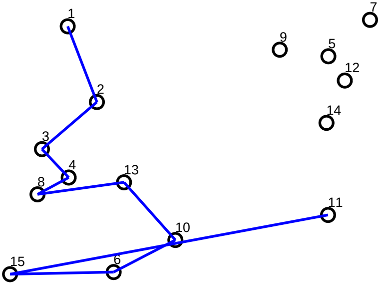

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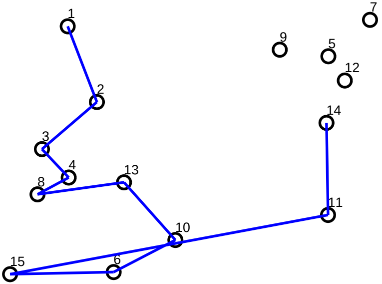

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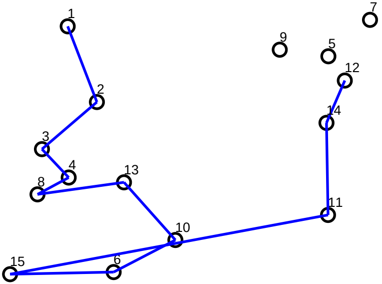

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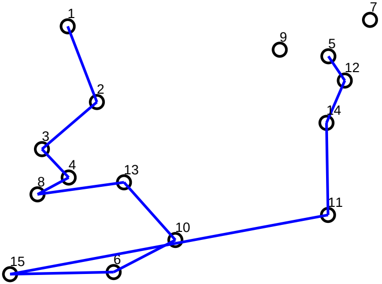

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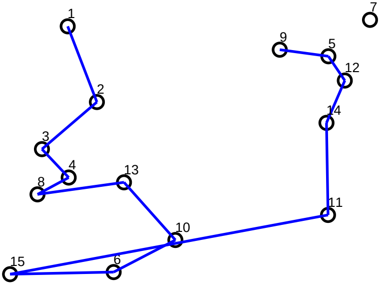

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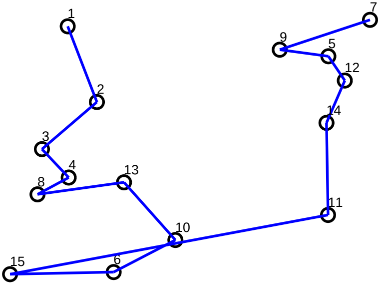

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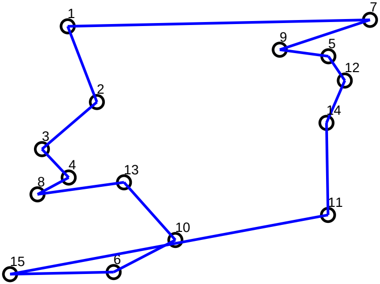

Improvement heuristics

Algorithms for improving a feasible solution. Local search heuristics. Utilize the fact that we are considering *metric* TSP problems.

Examples: *k-opt heuristics*, *crossing elimination*

Relaxation algorithms

Gives lower bounds on objective value. Trick is to relax the problem such that

- the reduced problem is "easy" to solve, and
- the lower bound given by the relaxation is "fairly" good.

Tradeoff between the two objectives.
Examples: *Branch and bound*, *Cutting plane methods*, *1-tree Lagrangian relaxation*.

1-tree Lagrangian relaxation

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- Lagrangian relax the assignment constraints

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  for all nodes except one, say node s. This means assigning a Lagrangian multiplier to each node (node price).

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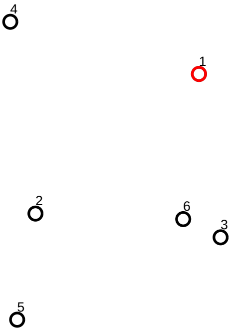
- Resulting problem is a *1-MST problem*, which is the problem of finding a minimum spanning tree on the nodes $\mathcal{N} \setminus \{s\}$, and then connecting node s to the tree by two links.
- Iteratively updating the node prices such that we increase our lower bound in each iteration.

1-tree Lagrangian relaxation



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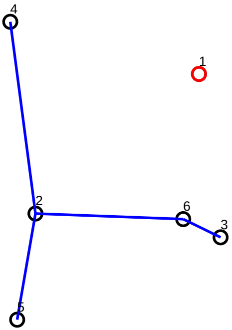

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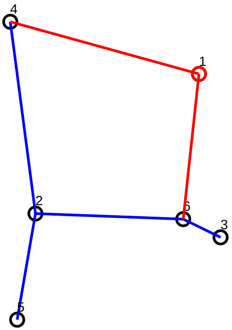

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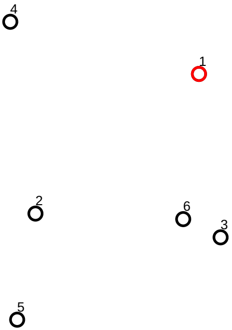

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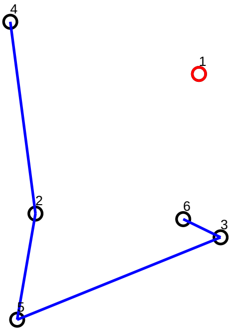

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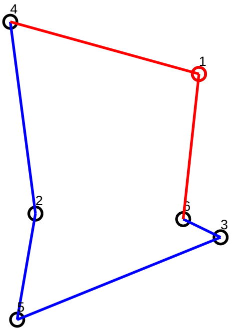

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- Get familiar with one of the most studied problems in optimization.
- Use CPLEX to solve some small problems.
- Develop and implement different algorithms, both heuristics and relaxation algorithms.
- Getting more familiar with *either* the theory of relaxation algorithms *or* probabilistic heuristics.

Thank you for listening.
Questions?