

MVE165/MMG630, Applied Optimization
Lecture 2
**Convexity; basic feasible solutions; the
simplex method; degeneracy; unbounded
solutions**

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2011-03-24

Mathematical optimization models

$$\left[\begin{array}{ll} \text{minimize or maximize} & f(x_1, \dots, x_n) \\ \text{subject to} & g_i(x_1, \dots, x_n) \quad \left\{ \begin{array}{l} \leq \\ = \\ \geq \end{array} \right\} b_i, \quad i = 1, \dots, m \end{array} \right]$$

- x_1, \dots, x_n are the **decision variables**
- f and g_1, \dots, g_m are given **functions** of the decision variables
- b_1, \dots, b_m are specified **constant parameters**
- The functions can be nonlinear, e.g. quadratic, exponential, logarithmic, non-analytic, ...
- In general, linear forms are more tractable than non-linear

Linear optimization models (programs)

- The production inventory model is a **linear program** (LP), i.e., all relations are described by linear forms
- A general linear program:

$$\left[\begin{array}{ll} \text{min or max} & c_1x_1 + \dots + c_nx_n \\ \text{subject to} & a_{i1}x_1 + \dots + a_{in}x_n \quad \left\{ \begin{array}{l} \leq \\ = \\ \geq \end{array} \right\} b_i, \quad i = 1, \dots, m \\ & x_j \geq 0, \quad j = 1, \dots, n \end{array} \right]$$

- The non-negativity constraints on x_j , $j = 1, \dots, n$ are not necessary, but usually assumed (reformulation always possible)

Discrete/integer/binary modelling

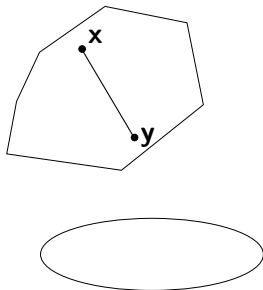
- A variable is called *discrete* if it can take only a countable set of values, e.g.,
 - Continuous variable: $x \in [0, 8]$ or $0 \leq x \leq 8$
 - Discrete variable: $x \in \{0, 4.4, 5.2, 8.0\}$
 - *Integer* variable: $x \in \{0, 1, 4, 5, 8\}$
- A *binary* variable can only take the values 0 or 1, i.e., all or nothing
E.g., a wind-mill can produce electricity only if it is built
 - Let $y = 1$ if the mill is built, otherwise $y = 0$
 - Capacity of a mill: C
 - Production $x \leq Cy$ (also limited by wind force etc.)
- In general, models with only continuous variables are more tractable than models with integrality/discrete requirements on the variables, but exceptions exist!

Convex sets

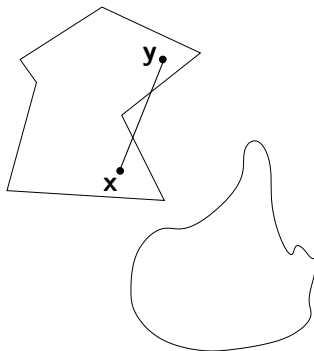
- A set S is **convex** if, for any elements $\mathbf{x}, \mathbf{y} \in S$ it holds that
$$\alpha \mathbf{x} + (1 - \alpha) \mathbf{y} \in S \text{ for all } 0 \leq \alpha \leq 1$$

- Examples:

Convex sets



Non-convex sets



\Rightarrow Intersections of linear (in)equalities \Rightarrow convex sets

Convex and concave functions

- A function f is **convex** on the set S if, for any elements $\mathbf{x}, \mathbf{y} \in S$ it holds that

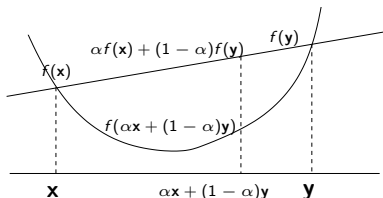
$$f(\alpha \mathbf{x} + (1 - \alpha) \mathbf{y}) \leq \alpha f(\mathbf{x}) + (1 - \alpha) f(\mathbf{y}) \text{ for all } 0 \leq \alpha \leq 1$$

- A function f is **concave** on the set S if, for any elements $\mathbf{x}, \mathbf{y} \in S$ it holds that

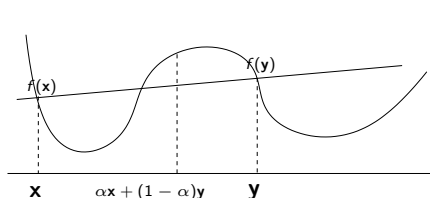
$$f(\alpha \mathbf{x} + (1 - \alpha) \mathbf{y}) \geq \alpha f(\mathbf{x}) + (1 - \alpha) f(\mathbf{y}) \text{ for all } 0 \leq \alpha \leq 1$$

⇒ Linear functions are convex (and concave)

Convex function



Non-convex function



Global solutions of convex optimization problem

- Let \mathbf{x}^* be a *local* minimizer of a *convex function* over a *convex set*. Then \mathbf{x}^* is also a *global* minimizer.
- ⇒ Every local optimum of a linear optimization problem is a global optimum
- If a linear optimization problem has any optimal solutions, at least one optimal solution is at an *extreme point* of the feasible set
- ⇒ Search for optimal extreme point(s)
- Next: *Linear* optimization problems and the *simplex method*

A general linear program – notation

minimize or maximize $c_1x_1 + \dots + c_nx_n$

subject to $a_{i1}x_1 + \dots + a_{in}x_n \left\{ \begin{array}{l} \leq \\ = \\ \geq \end{array} \right\} b_i, \quad i = 1, \dots, m$

$x_j \left\{ \begin{array}{l} \leq 0 \\ \text{unrestricted in sign} \\ \geq 0 \end{array} \right\}, \quad j = 1, \dots, n$

- c_j , a_{ij} , and b_i are constant parameters for $i = 1, \dots, m$ and $j = 1, \dots, n$

The standard form and the simplex method for linear programs

- Every linear program can be reformulated such that:
 - all constraints are expressed as equalities with non-negative right hand sides
 - all variables are restricted to be non-negative
- Referred to as the *standard form*
- These requirements streamline the calculations of the *simplex method*
- *Software solvers* (e.g., Cplex, GLPK, Clp) can handle also inequality constraints and unrestricted variables – the reformulations are made automatically

The simplex method—reformulations

- The lego example:

$$\begin{bmatrix} 2x_1 & +x_2 & \leq & 6 \\ 2x_1 & +2x_2 & \leq & 8 \\ & x_1, x_2 & \geq & 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 2x_1 & +x_2 & +s_1 & = & 6 \\ 2x_1 & +2x_2 & & +s_2 & = & 8 \\ & x_1, x_2, s_1, s_2 & \geq & 0 \end{bmatrix}$$

- s_1 and s_2 are called *slack variables*—they “fill out” the (positive) distances between the left and right hand sides
- *Surplus variable* s_3 (a different example):

$$\begin{bmatrix} x_1 & + & x_2 & \geq & 800 \\ & x_1, x_2 & \geq & 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} x_1 & + & x_2 & - & s_3 & = & 800 \\ & x_1, x_2, s_3 & \geq & 0 \end{bmatrix}$$

The simplex method—reformulations, cont.

- Non-negative right hand side:

$$\begin{bmatrix} x_1 - x_2 \leq -23 \\ x_1, x_2 \geq 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} -x_1 + x_2 \geq 23 \\ x_1, x_2 \geq 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} -x_1 + x_2 - s_4 = 23 \\ x_1, x_2, s_4 \geq 0 \end{bmatrix}$$

- Sign-restricted (non-negative) variables:

$$\begin{bmatrix} x_1 + x_2 \leq 10 \\ x_1 \geq 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} x_1 + x_2^1 - x_2^2 \leq 10 \\ x_1, x_2^1, x_2^2 \geq 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} x_1 + x_2^1 - x_2^2 + s_5 = 10 \\ x_1, x_2^1, x_2^2, s_5 \geq 0 \end{bmatrix}$$

Basic feasible solutions

- Consider m equations of n variables, where $m \leq n$
- Set $n - m$ variables to zero and solve (if possible) the remaining $(m \times m)$ system of equations
- If the solution is *unique*, it is called a *basic* solution
- A basic solution corresponds to an *blue intersection* (feasible ($x \geq 0$) or infeasible ($x \not\geq 0$)) of m hyperplanes in \mathbb{R}^m
- Each *extreme point* of the feasible set is an intersection of m hyperplanes such that all variable values are ≥ 0
- **Basic feasible solution \Leftrightarrow extreme point of the feasible set**

$$\begin{array}{rcl} a_{11}x_1 + \dots + a_{1n}x_n = b_1 & & x_1 \geq 0 \\ a_{21}x_1 + \dots + a_{2n}x_n = b_2 & & x_2 \geq 0 \\ & \dots & \dots \\ a_{m1}x_1 + \dots + a_{mn}x_n = b_m & & x_n \geq 0 \end{array}$$

Basic feasible solutions, example

- Constraints:

$$x_1 \leq 23 \quad (1)$$

$$0.067x_1 + x_2 \leq 6 \quad (2)$$

$$3x_1 + 8x_2 \leq 85 \quad (3)$$

$$x_1, x_2 \geq 0$$

- Add slack variables:

$$x_1 + s_1 = 23 \quad (1)$$

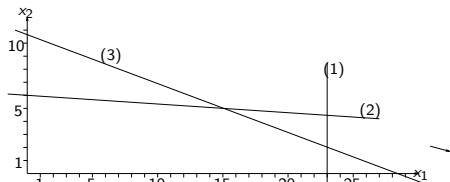
$$0.067x_1 + x_2 + s_2 = 6 \quad (2)$$

$$3x_1 + 8x_2 + s_3 = 85 \quad (3)$$

$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

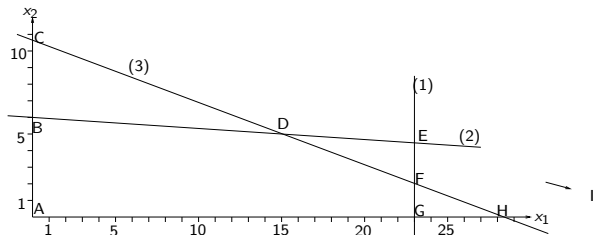
$$m = 3$$

$$n = 5$$

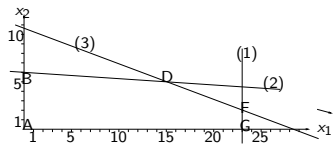


Basic and non-basic variables and solutions

basic variables	basic solution			non-basic variables (0, 0)	point	feasible?
s_1, s_2, s_3	23	6	85	x_1, x_2	A	yes
s_1, s_2, x_1	$-5\frac{1}{3}$	$4\frac{1}{9}$	$28\frac{1}{3}$	s_3, x_2	H	no
s_1, s_2, x_2	23	$-4\frac{5}{8}$	$10\frac{5}{8}$	x_1, s_3	C	no
s_1, x_1, s_3	-67	90	-185	s_2, x_2	I	no
s_1, x_2, s_3	23	6	37	s_2, x_1	B	yes
x_1, s_2, s_3	23	$4\frac{7}{15}$	16	s_1, x_2	G	yes
x_2, s_2, s_3	-	-	-	s_1, x_1	-	-
x_1, x_2, s_1	15	5	8	s_2, s_3	D	yes
x_1, x_2, s_2	23	2	$2\frac{7}{15}$	s_1, s_3	F	yes
x_1, x_2, s_3	23	$4\frac{7}{15}$	$-19\frac{11}{15}$	s_1, s_2	E	no



Basic feasible solutions correspond to solutions to the system of equations that fulfil non-negativity



$$\begin{bmatrix} x_1 & +s_1 & & = 23 \\ 0.067x_1 & +x_2 & +s_2 & = 6 \\ 3x_1 & +8x_2 & & +s_3 = 85 \end{bmatrix}$$

$$A: x_1 = x_2 = 0 \Rightarrow \begin{bmatrix} s_1 & & = 23 \\ & s_2 & = 6 \\ & & s_3 = 85 \end{bmatrix}$$

$$B: x_1 = s_2 = 0 \Rightarrow \begin{bmatrix} & s_1 & = 23 \\ x_2 & & = 6 \\ 8x_2 & +s_3 & = 85 \end{bmatrix}$$

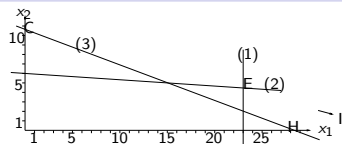
$$D: s_3 = s_2 = 0 \Rightarrow \begin{bmatrix} x_1 & +s_1 & = 23 \\ 0.067x_1 & +x_2 & = 6 \\ 3x_1 & +8x_2 & = 85 \end{bmatrix}$$

$$F: s_3 = s_1 = 0 \Rightarrow \begin{bmatrix} x_1 & & = 23 \\ 0.067x_1 & +x_2 & +s_2 = 6 \\ 3x_1 & +8x_2 & = 85 \end{bmatrix}$$

$$G: x_2 = s_1 = 0 \Rightarrow \begin{bmatrix} x_1 & & = 23 \\ 0.067x_1 & +s_2 & = 6 \\ 3x_1 & & +s_3 = 85 \end{bmatrix}$$

Basic infeasible solutions correspond to solutions to the system of equations with one or more variables

< 0



$$\begin{bmatrix} x_1 + s_1 = 23 \\ 0.067x_1 + x_2 + s_2 = 6 \\ 3x_1 + 8x_2 + s_3 = 85 \end{bmatrix}$$

$$\text{H: } x_2 = s_3 = 0 \Rightarrow \begin{bmatrix} x_1 + s_1 = 23 \\ 0.067x_1 + s_2 = 6 \\ 3x_1 = 85 \end{bmatrix}$$

$$\text{C: } x_1 = s_3 = 0 \Rightarrow \begin{bmatrix} s_1 = 23 \\ x_2 + s_2 = 6 \\ 8x_2 = 85 \end{bmatrix}$$

$$\text{I: } s_2 = x_2 = 0 \Rightarrow \begin{bmatrix} x_1 + s_1 = 23 \\ 0.067x_1 = 6 \\ 3x_1 + s_3 = 85 \end{bmatrix}$$

$$\text{-: } s_1 = x_1 = 0 \Rightarrow \begin{bmatrix} 0 = 23 \\ x_2 + s_2 = 6 \\ 8x_2 + s_3 = 85 \end{bmatrix}$$

$$\text{E: } s_1 = s_2 = 0 \Rightarrow \begin{bmatrix} x_1 = 23 \\ 0.067x_1 + x_2 = 6 \\ 3x_1 + 8x_2 + s_3 = 85 \end{bmatrix}$$

Basic feasible solutions and the simplex method

- Express the m *basic* variables in terms of the $n - m$ *non-basic* variables
- Example: Start at $x_1 = x_2 = 0 \Rightarrow s_1, s_2, s_3$ are *basic*

$$\begin{bmatrix} x_1 & & +s_1 & & = 23 \\ \frac{1}{15}x_1 & +x_2 & & +s_2 & = 6 \\ 3x_1 & +8x_2 & & & +s_3 = 85 \end{bmatrix}$$

- Express $s_1, s_2,$ and s_3 in terms of x_1 and x_2 (*non-basic*):

$$\begin{bmatrix} s_1 = 23 & -x_1 \\ s_2 = 6 & -\frac{1}{15}x_1 & -x_2 \\ s_3 = 85 & -3x_1 & -8x_2 \end{bmatrix}$$

- Express the objective in terms of the *non-basic* variables:

$$z = 2x_1 + 3x_2 \quad \Leftrightarrow \quad z - 2x_1 - 3x_2 = 0$$

Basic feasible solutions and the simplex method

- The *first basic solution* can be represented as

$-z$	$+2x_1$	$+3x_2$		$= 0$	(0)
	x_1		$+s_1$	$= 23$	(1)
	$\frac{1}{15}x_1$	$+x_2$		$+s_2 = 6$	(2)
	$3x_1$	$+8x_2$		$+s_3 = 85$	(3)

- Marginal values** for increasing the non-basic variables x_1 and x_2 from zero: 2 and 3, resp.

⇒ Choose x_2 — let x_2 *enter the basis* DRAW GRAPH!!

- One basic variable (s_1 , s_2 , or s_3) must *leave the basis*. Which?
- The value of x_2 can increase until some basic variable reaches the value 0:

$$\left. \begin{array}{l}
 (2) : s_2 = 6 - x_2 \geq 0 \quad \Rightarrow x_2 \leq 6 \\
 (3) : s_3 = 85 - 8x_2 \geq 0 \quad \Rightarrow x_2 \leq 10\frac{5}{8}
 \end{array} \right\} \Rightarrow \begin{array}{l}
 s_2 = 0 \text{ when} \\
 x_2 = 6 \\
 (\text{and } s_3 = 37)
 \end{array}$$

- s_2 will leave the basis

Change basis through row operations

- Eliminate s_2 from the basis, let x_2 enter the basis using row operations:

$-z$	$+2x_1$	$+3x_2$			$=$	0	(0)
	x_1		$+s_1$		$=$	23	(1)
	$\frac{1}{15}x_1$	$+x_2$		$+s_2$	$=$	6	(2)
	$3x_1$	$+8x_2$		$+s_3$	$=$	85	(3)
$-z$	$+\frac{9}{5}x_1$			$-3s_2$	$=$	-18	$(0) -3 \cdot (2)$
	x_1		$+s_1$		$=$	23	$(1) - 0 \cdot (2)$
	$\frac{1}{15}x_1$	$+x_2$		$+s_2$	$=$	6	(2)
	$\frac{37}{15}x_1$			$-8s_2 + s_3$	$=$	37	$(3) - 8 \cdot (2)$

- Corresponding basic solution: $s_1 = 23$, $x_2 = 6$, $s_3 = 37$.
- Nonbasic variables: $x_1 = s_2 = 0$
- The marginal value of x_1 is $\frac{9}{5} > 0$. Let x_1 enter the basis
- Which one should leave? s_1 , x_2 , or s_3 ?

Change basis ...

$-z$	$+\frac{9}{5}x_1$		$-3s_2$	$=$	-18	(0)
	x_1		$+s_1$	$=$	23	(1)
	$\frac{1}{15}x_1$	$+x_2$	$+s_2$	$=$	6	(2)
	$\frac{37}{15}x_1$		$-8s_2 + s_3$	$=$	37	(3)

- The value of x_1 can increase until some basic variable reaches the value 0:

$$\left. \begin{array}{l} (1) : s_1 = 23 - x_1 \geq 0 \Rightarrow x_1 \leq 23 \\ (2) : x_2 = 6 - \frac{1}{15}x_1 \geq 0 \Rightarrow x_1 \leq 90 \\ (3) : s_3 = 37 - \frac{37}{15}x_1 \geq 0 \Rightarrow x_1 \leq 15 \end{array} \right\} \Rightarrow \begin{array}{l} s_3 = 0 \text{ when} \\ x_1 = 15 \end{array}$$

- x_1 enters the basis and s_3 leaves the basis
- Perform row operations:

$-z$		$+2.84s_2$	$-0.73s_3$	$=$	-45	$(0) - (3) \cdot \frac{15}{37} \cdot \frac{9}{5}$
	s_1	$+3.24s_2$	$-0.41s_3$	$=$	8	$(1) - (3) \cdot \frac{15}{37}$
	x_2	$+1.22s_2$	$-0.03s_3$	$=$	5	$(2) - (3) \cdot \frac{15}{37} \cdot \frac{1}{15}$
	x_1	$-3.24s_2$	$+0.41s_3$	$=$	15	$(3) \cdot \frac{15}{37}$

Change basis ...

$-z$		$+2.84s_2$	$-0.73s_3$	$=$	-45	(0)
	s_1	$+3.24s_2$	$-0.41s_3$	$=$	8	(1)
	x_2	$+1.22s_2$	$-0.03s_3$	$=$	5	(2)
	x_1	$-3.24s_2$	$+0.41s_3$	$=$	15	(3)

- Let s_2 enter the basis (marginal value > 0)
- The value of s_2 can increase until some basic variable = 0:

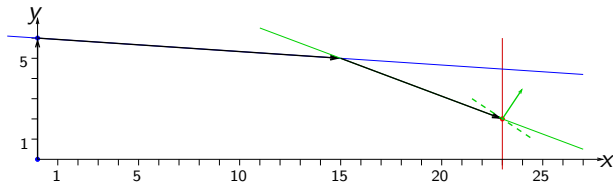
$$\left. \begin{array}{l} (1) : s_1 = 8 - 3.24s_2 \geq 0 \Rightarrow s_2 \leq 2.47 \\ (2) : x_2 = 5 - 1.22s_2 \geq 0 \Rightarrow s_2 \leq 4.10 \\ (3) : x_1 = 15 + 3.24s_2 \geq 0 \Rightarrow s_2 \geq -4.63 \end{array} \right\} \Rightarrow \begin{array}{l} s_1 = 0 \text{ when} \\ s_2 = 2.47 \end{array}$$
- s_2 enters the basis and s_1 will leave the basis
- Perform row operations:

$-z$		$-0.87s_1$		$-0.37s_3$	$=$	-52	$(0) - (1) \cdot \frac{2.84}{3.24}$
		$0.31s_1$	$+s_2$	$-0.12s_3$	$=$	2.47	$(1) \cdot \frac{1}{3.24}$
	x_2	$-0.37s_1$		$+0.12s_3$	$=$	2	$(2) - (1) \cdot \frac{1.22}{3.24}$
	x_1	$+s_1$			$=$	23	$(3) + (1)$

Optimal basic solution

$-z$	$-0.87s_1$	$-0.37s_3$	$=$	-52	
	$0.31s_1$	$+s_2$	$-0.12s_3$	$=$	2.47
	x_2	$-0.37s_1$	$+0.12s_3$	$=$	2
	x_1	$+s_1$		$=$	23

- No marginal value is positive. No improvement can be made
- The optimal basis is given by $s_2 = 2.47$, $x_2 = 2$, and $x_1 = 23$
- Non-basic variables: $s_1 = s_3 = 0$
- Optimal value: $z = 52$



Summary of the solution course

basis	$-z$	x_1	x_2	s_1	s_2	s_3	RHS
$-z$	1	2	3	0	0	0	0
s_1	0	1	0	1	0	0	23
s_2	0	0.067	1	0	1	0	6
s_3	0	3	8	0	0	1	85
$-z$	1	1.80	0	0	-3	0	-18
s_1	0	1	0	1	0	0	23
x_2	0	0.07	1	0	1	0	6
s_3	0	2.47	0	0	-8	1	37
$-z$	1	0	0	0	2.84	-0.73	-45
s_1	0	0	0	1	3.24	-0.41	8
x_2	0	0	1	0	1.22	-0.03	5
x_1	0	1	0	0	-3.24	0.41	15
$-z$	1	0	0	-0.87	0	-0.37	-52
s_2	0	0	0	0.31	1	-0.12	2.47
x_2	0	0	1	-0.37	0	0.12	2
x_1	0	1	0	1	0	0	23

Summary of the simplex method

- **Optimality condition:** The *entering* variable in a maximization (minimization) problem should have the largest positive (negative) marginal value (reduced cost).

The entering variable *determines a direction* in which the objective value increases (decreases).

If all *reduced costs are negative* (positive), the current basis is *optimal*.

- **Feasibility condition:** The *leaving* variable is the one with smallest nonnegative quotient.

Corresponds to the constraint that is “reached first”

Simplex search for linear (minimization) programs (Ch. 4.6)

- 1 **Initialization:** Choose any feasible basis, construct the corresponding basic solution \mathbf{x}^0 , let $t = 0$
- 2 **Step direction:** Select a variable to enter the basis using the optimality condition (negative marginal value). Stop if no entering variable exists
- 3 **Step length:** Select a leaving variable using the feasibility condition (smallest non-negative quotient)
- 4 **New iterate:** Compute the new basic solution \mathbf{x}^{t+1} by performing matrix operations.
- 5 Let $t := t + 1$ and repeat from 2

Solve the lego problem using the simplex method!

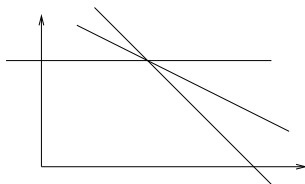
$$\begin{array}{rcll} \text{maximize} & z = & 1600x_1 & + & 1000x_2 \\ \text{subject to} & & 2x_1 & + & x_2 \leq 6 \\ & & 2x_1 & + & 2x_2 \leq 8 \\ & & & & x_1, x_2 \geq 0 \end{array}$$

HOMEWORK!!

Degeneracy (Ch. 4.10)

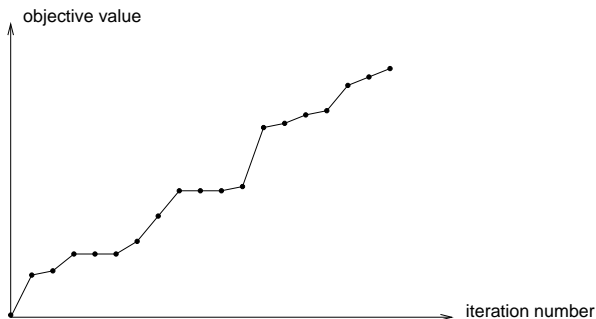
- If the smallest nonnegative quotient is zero, the value of a basic variable will become zero in the next iteration
- The solution is *degenerate*
- The objective value will *not* improve in this iteration
- Risk: *cycling* around (non-optimal) bases
- Reason: a *redundant* constraint “touches” the feasible set
- Example:

$$\begin{array}{rcll} x_1 & + & x_2 & \leq 6 \\ & & x_2 & \leq 3 \\ x_1 & + & 2x_2 & \leq 9 \\ & & x_1, x_2 & \geq 0 \end{array}$$



Degeneracy

- Typical objective function progress of the simplex method



- Computation rules to prevent from infinite cycling: careful choices of leaving and entering variables
- In modern software: perturb the right hand side ($b_i + \Delta b_i$), solve, reduce the perturbation and resolve starting from the current basis. Repeat until $\Delta b_i = 0$.

Unbounded solutions (Ch. 4.4, 4.6)

- If all quotients are *negative*, the value of the variable entering the basis may increase *infinitely*
- The feasible set is *unbounded*
- In a real application this would probably be due to some incorrect assumption

• Example:

$$\begin{array}{ll} \text{minimize} & z = -x_1 - 2x_2 \\ \text{subject to} & -x_1 + x_2 \leq 2 \\ & -2x_1 + x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{array}$$

DRAW GRAPH!!

Unbounded solutions (Ch. 4.4, 4.6)

- A feasible basis is given by $x_1 = 1$, $x_2 = 3$, with corresponding tableau:

Homework: Find this basis using the simplex method.

basis	$-z$	x_1	x_2	s_1	s_2	RHS
$-z$	1	0	0	5	-3	7
x_1	0	1	0	1	-1	1
x_2	0	0	1	2	-1	3

- Entering variable is s_2
- Row 1: $x_1 = 1 + s_2 \geq 0 \Rightarrow s_2 \geq -1$
- Row 2: $x_2 = 3 + s_2 \geq 0 \Rightarrow s_2 \geq -3$
- No leaving variable can be found, since no constraint will prevent s_2 from increasing infinitely