

**MVE165/MMG630, Applied Optimization**  
**Lecture 3**  
**Infeasibility; starting solutions; duality;**  
**interpretation**

Ann-Brith Strömberg

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# Starting solution—finding an initial basis (Ch. 4.9)

- ▶ Example:

$$\begin{array}{ll} \text{minimize} & z = 2x_1 + 3x_2 \\ \text{subject to} & 3x_1 + 2x_2 = 14 \\ & 2x_1 - 4x_2 \geq 2 \\ & 4x_1 + 3x_2 \leq 19 \\ & x_1, x_2 \geq 0 \end{array}$$

DRAW GRAPH!!

- ▶ Add slack and surplus variables

$$\begin{array}{ll} \text{minimize} & z = 2x_1 + 3x_2 \\ \text{subject to} & 3x_1 + 2x_2 = 14 \\ & 2x_1 - 4x_2 - s_1 = 2 \\ & 4x_1 + 3x_2 + s_2 = 19 \\ & x_1, x_2, s_1, s_2 \geq 0 \end{array}$$

- ▶ How finding an initial basis? Only  $s_2$  is obvious!

# Artificial variables

- ▶ Add artificial variables  $a_1$  and  $a_2$  to the first and second constraints, respectively
- ▶ Solve an artificial problem: minimize  $a_1 + a_2$

$$\begin{array}{llllllll} \text{minimize} & w = & & & & a_1 & + & a_2 \\ \text{subject to} & & 3x_1 & + & 2x_2 & & + & a_1 & = & 14 \\ & & 2x_1 & - & 4x_2 & - & s_1 & & + & a_2 & = & 2 \\ & & 4x_1 & + & 3x_2 & & & + & s_2 & & = & 19 \\ & & & & & & & & & & & & x_1, x_2, s_1, s_2, a_1, a_2 & \geq & 0 \end{array}$$

- ▶ The “phase one” problem
- ▶ An initial basis is given by  $a_1 = 14$ ,  $a_2 = 2$ , and  $s_2 = 19$ :

basis	$-w$	$x_1$	$x_2$	$s_1$	$s_2$	$a_1$	$a_2$	RHS
$-w$	1	-5	2	1	0	0	0	-16
$a_1$	0	3	2	0	0	1	0	14
$a_2$	0	2	-4	-1	0	0	1	2
$s_2$	0	4	3	0	1	0	0	19

# Find an initial solution using artificial variables

- $x_1$  enters  $\Rightarrow a_2$  leaves (then  $x_2 \Rightarrow s_2$ , then  $s_1 \Rightarrow a_1$ )

basis	$-w$	$x_1$	$x_2$	$s_1$	$s_2$	$a_1$	$a_2$	RHS
$-w$	1	-5	2	1	0	0	0	-16
$a_1$	0	3	2	0	0	1	0	14
$a_2$	0	2	-4	-1	0	0	1	2
$s_2$	0	4	3	0	1	0	0	19

$-w$	1	0	-8	-1.5	0	0		-11
$a_1$	0	0	8	1.5	0	1		11
$x_1$	0	1	-2	-0.5	0	0		1
$s_2$	0	0	11	2	1	0		15

$-w$	1	0	0	-0.045	0.727	0		-0.091
$a_1$	0	0	0	0.045	-0.727	1		0.091
$x_1$	0	1	0	-0.136	0.182	0		3.727
$x_2$	0	0	1	0.182	0.091	0		1.364

$-w$	1	0	0	0	0			0
$s_1$	0	0	0	1	-16			2
$x_1$	0	1	0	0	-2			4
$x_2$	0	0	1	0	3			1

- A feasible basis is given by  $x_1 = 4$ ,  $x_2 = 1$ , and  $s_1 = 2$

## Infeasible linear programs (Ch. 4.9)

- ▶ If the solution to the “phase one” problem has optimal value  $= 0$ , a feasible basis has been found
  - ⇒ Start optimizing the original objective function  $z$  from this basis (*homework*)
  - ▶ If the solution to the “phase one” problem has optimal value  $w > 0$ , no feasible solutions exist
  - ▶ What would this mean in a real application?
  - ▶ Alternative: Big- $M$  method: Add the artificial variables to the original objective—with a large coefficient
- Example:

$$\text{minimize } z = 2x_1 + 3x_2$$

$$\Rightarrow \text{minimize } z_a = 2x_1 + 3x_2 + Ma_1 + Ma_2$$

# Alternative optimal solutions (Ch. 4.6)

- ▶ Example:

$$\begin{array}{ll} \text{maximize} & z = 2x_1 + 4x_2 \\ \text{subject to} & x_1 + 2x_2 \leq 5 \\ & x_1 + x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{array}$$

DRAW GRAPH!!

- ▶ The extreme points  $(0, \frac{5}{2})$  and  $(3, 1)$  have the same optimal value  $z = 10$
- ▶ All solutions that are positive linear (convex) combinations of these are optimal:

$$(x_1, x_2) = \alpha \cdot (0, \frac{5}{2}) + (1 - \alpha) \cdot (3, 1), \quad 0 \leq \alpha \leq 1$$

# A general linear program in standard form

- ▶ A linear program with  $n$  non-negative variables,  $m$  equality constraints ( $m < n$ ), and non-negative right hand sides:

$$\begin{aligned} \text{maximize} \quad & z = \sum_{j=1}^n c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^n a_{ij} x_j = b_i, \quad i = 1, \dots, m, \\ & x_j \geq 0, \quad j = 1, \dots, n. \end{aligned}$$

- ▶ On matrix form it is written as:

$$\begin{aligned} \text{maximize} \quad & z = \mathbf{c}^T \mathbf{x}, \\ \text{subject to} \quad & \mathbf{Ax} = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}^n, \end{aligned}$$

where  $\mathbf{x} \in \mathbb{R}^n$ ,  $\mathbf{A} \in \mathbb{R}^{m \times n}$ ,  $\mathbf{b} \in \mathbb{R}_+^m$  ( $\mathbf{b} \geq \mathbf{0}^m$ ), and  $\mathbf{c} \in \mathbb{R}^n$ .

# General derivation of the simplex method (Ch. 4.8)

- ▶  $B$  = set of basic variables,  $N$  = set of non-basic variables
- ⇒  $|B| = m$  and  $|N| = n - m$
- ▶ Partition matrix/vectors:  $\mathbf{A} = (\mathbf{B}, \mathbf{N})$ ,  $\mathbf{x} = (\mathbf{x}_B, \mathbf{x}_N)$ ,  $\mathbf{c} = (\mathbf{c}_B, \mathbf{c}_N)$
- ▶ The matrix  $\mathbf{B}$  ( $\mathbf{N}$ ) contains the columns of  $\mathbf{A}$  corresponding to the index set  $B$  ( $N$ ) — Analogously for  $\mathbf{x}$  and  $\mathbf{c}$
- ▶ Rewrite the linear program:

$$\left[ \begin{array}{l} \text{maximize } z = \mathbf{c}^T \mathbf{x} \\ \text{subject to } \mathbf{A}\mathbf{x} = \mathbf{b}, \\ \mathbf{x} \geq \mathbf{0}^n \end{array} \right] = \left[ \begin{array}{l} \text{maximize } z = \mathbf{c}_B^T \mathbf{x}_B + \mathbf{c}_N^T \mathbf{x}_N \\ \text{subject to } \mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_N = \mathbf{b}, \\ \mathbf{x}_B \geq \mathbf{0}^m, \mathbf{x}_N \geq \mathbf{0}^{n-m} \end{array} \right]$$

- ▶ Substitute:  $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N \implies$

$$\begin{array}{ll} \text{maximize} & z = \mathbf{c}_B^T \mathbf{B}^{-1}\mathbf{b} + [\mathbf{c}_N^T - \mathbf{c}_B^T \mathbf{B}^{-1}\mathbf{N}]\mathbf{x}_N \\ \text{subject to} & \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N \geq \mathbf{0}^m, \\ & \mathbf{x}_N \geq \mathbf{0}^{n-m} \end{array}$$



# Optimality and feasibility

- ▶ **Optimality condition** (for maximization)

The basis  $B$  is optimal if  $\mathbf{c}_N^T - \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{N} \leq \mathbf{0}^{n-m}$   
(marginal values = reduced costs  $\leq 0$ )

If not, choose as entering variable  $j \in N$  the one with the largest value of the reduced cost  $c_j - \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{A}_j$

- ▶ **Feasibility condition**

For all  $i \in B$  it holds that  $x_i = (\mathbf{B}^{-1} \mathbf{b})_i - (\mathbf{B}^{-1} \mathbf{A}_j)_i x_j$

Choose the leaving variable  $i^* \in B$  according to

$$i^* = \arg \min_{i \in B} \left\{ \frac{(\mathbf{B}^{-1} \mathbf{b})_i}{(\mathbf{B}^{-1} \mathbf{A}_j)_i} \mid (\mathbf{B}^{-1} \mathbf{A}_j)_i > 0 \right\}$$

# In the simplex tableau, we have

basis	$-z$	$\mathbf{x}_B$	$\mathbf{x}_N$	$\mathbf{s}$	RHS
$-z$	1	$\mathbf{0}$	$\mathbf{c}_N^T - \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{N}$	$-\mathbf{c}_B^T \mathbf{B}^{-1}$	$-\mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{b}$
$\mathbf{x}_B$	$\mathbf{0}$	$\mathbf{I}$	$\mathbf{B}^{-1} \mathbf{N}$	$\mathbf{B}^{-1}$	$\mathbf{B}^{-1} \mathbf{b}$

- ▶  $\mathbf{s}$  denotes possible slack variables (columns for  $\mathbf{s}$  are copies of certain columns for  $(\mathbf{x}_B, \mathbf{x}_N)$ )
  - ▶ The computations performed by the simplex algorithm involve matrix inversions and updates of these
  - ▶ A non-basic (basic) variable enters (leaves) the basis  $\Rightarrow$  one column,  $\mathbf{A}_j$ , of  $\mathbf{B}$  is replaced by another,  $\mathbf{A}_k$
  - ▶ Row operations  $\Leftrightarrow$  Updates of  $\mathbf{B}^{-1}$  (and  $\mathbf{B}^{-1} \mathbf{N}$ ,  $\mathbf{B}^{-1} \mathbf{b}$ , and  $\mathbf{c}_B^T \mathbf{B}^{-1}$ )
- $\Rightarrow$  Efficient numerical computations are crucial for the performance of the simplex algorithm

# “Intuitive” derivation of duality (Ch. 6.1)

- ▶ A linear program with optimal value  $z^*$

$$\begin{array}{llll} \text{maximize} & z := & 20x_1 & +18x_2 & & \text{weights} \\ \text{subject to} & & 7x_1 & +10x_2 & \leq 3600 & (1) & v_1 \\ & & 16x_1 & +12x_2 & \leq 5400 & (2) & v_2 \\ & & & & x_1, x_2 & \geq 0 & \end{array}$$

- ▶ How large can  $z^*$  be?
- ▶ Compute upper estimates of  $z^*$ , e.g.
  - ▶ Multiply (1) by 3  $\Rightarrow 21x_1 + 30x_2 \leq 10800 \Rightarrow z^* \leq 10800$
  - ▶ Multiply (2) by 1.5  $\Rightarrow 24x_1 + 18x_2 \leq 8100 \Rightarrow z^* \leq 8100$
  - ▶ Combine:  $0.6 \times (1) + 1 \times (2) \Rightarrow 20.2x_1 + 18x_2 \leq 7560 \Rightarrow z^* \leq 7560$
- ▶ Do better than guess—compute optimal weights!
- ▶ Value of estimate:  $w = 3600v_1 + 5400v_2 \rightarrow \min$
- ▶ Constraints on weights: 
$$\left[ \begin{array}{ll} 7v_1 + 16v_2 & \geq 20 \\ 10v_1 + 12v_2 & \geq 18 \\ v_1, v_2 & \geq 0 \end{array} \right]$$

# The best (lowest) possible upper estimate of $z^*$

$$\begin{array}{ll} \text{minimize} & w := 3600v_1 + 5400v_2 \\ \text{subject to} & 7v_1 + 16v_2 \geq 20 \\ & 10v_1 + 12v_2 \geq 18 \\ & v_1, v_2 \geq 0 \end{array}$$

- ▶ A linear program!
- ▶ It is called the **dual** of the original linear program

# The lego model – the market problem

- ▶ Consider the lego problem

$$\begin{array}{ll} \text{maximize } z = & 1600x_1 + 1000x_2 \\ \text{subject to} & 2x_1 + x_2 \leq 6 \\ & 2x_1 + 2x_2 \leq 8 \\ & x_1, x_2 \geq 0 \end{array}$$

- ▶ Option: Sell bricks instead of making furniture
- ▶  $v_1(v_2)$  = price of a large (small) brick
- ▶ Market wish to minimize payment: *minimize*  $6v_1 + 8v_2$
- ▶ I sell if prices are high enough:
  - ▶  $2v_1 + 2v_2 \geq 1600$  – otherwise better to make tables
  - ▶  $v_1 + 2v_2 \geq 1000$  – otherwise better to make chairs
  - ▶  $v_1, v_2 \geq 0$  – prices are naturally non-negative

# Linear programming duality

- ▶ To each primal linear program corresponds a dual linear program

$$\begin{aligned} \text{[Primal]} \quad & \text{minimize} && z = \mathbf{c}^T \mathbf{x}, \\ & \text{subject to} && \mathbf{Ax} = \mathbf{b}, \\ & && \mathbf{x} \geq \mathbf{0}^n, \end{aligned}$$

$$\begin{aligned} \text{[Dual]} \quad & \text{maximize} && w = \mathbf{b}^T \mathbf{y}, \\ & \text{subject to} && \mathbf{A}^T \mathbf{y} \leq \mathbf{c}. \end{aligned}$$

- ▶ On component form:

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$$\begin{aligned} \text{[Primal]} \quad & \text{minimize} && z = \sum_{j=1}^n c_j x_j \\ & \text{subject to} && \sum_{j=1}^n a_{ij} x_j = b_i, \quad i = 1, \dots, m, \\ & && x_j \geq 0, \quad j = 1, \dots, n, \end{aligned}$$

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$$\begin{aligned} \text{[Dual]} \quad & \text{maximize} && w = \sum_{i=1}^m b_i y_i \\ & \text{subject to} && \sum_{i=1}^m a_{ij} y_i \leq c_j, \quad j = 1, \dots, n. \end{aligned}$$

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# An example

- ▶ A primal linear program

$$\begin{array}{ll} \text{minimize} & z = 2x_1 + 3x_2 \\ \text{subject to} & 3x_1 + 2x_2 = 14 \\ & 2x_1 - 4x_2 \geq 2 \\ & 4x_1 + 3x_2 \leq 19 \\ & x_1, x_2 \geq 0 \end{array}$$

- ▶ The corresponding dual linear program

$$\begin{array}{ll} \text{maximize} & w = 14y_1 + 2y_2 + 19y_3 \\ \text{subject to} & 3y_1 + 2y_2 + 4y_3 \leq 2 \\ & 2y_1 - 4y_2 + 3y_3 \leq 3 \\ & y_1 \text{ free,} \\ & y_2 \geq 0, \\ & y_3 \leq 0 \end{array}$$

# Rules for constructing the dual program (Ch. 6.2)

maximization	$\Leftrightarrow$	minimization
dual program	$\Leftrightarrow$	primal program
primal program	$\Leftrightarrow$	dual program
<i>constraints</i>		<i>variables</i>
$\geq$	$\Leftrightarrow$	$\leq 0$
$\leq$	$\Leftrightarrow$	$\geq 0$
$=$	$\Leftrightarrow$	free
<i>variables</i>		<i>constraints</i>
$\geq 0$	$\Leftrightarrow$	$\geq$
$\leq 0$	$\Leftrightarrow$	$\leq$
free	$\Leftrightarrow$	$=$

The dual of the dual of any linear program equals the primal



## Duality properties (Ch. 6.3)

- ▶ **Weak duality:** Let  $\mathbf{x}$  be a feasible point in the primal (minimization) and  $\mathbf{y}$  be a feasible point in the dual (maximization). Then,

$$z = \mathbf{c}^T \mathbf{x} \geq \mathbf{b}^T \mathbf{y} = w$$

- ▶ **Strong duality:** In a pair of primal and dual linear programs, if one of them has an optimal solution, so does the other, and their optimal values are equal.
- ▶ **Complementary slackness:** If  $\mathbf{x}$  is optimal in the primal and  $\mathbf{y}$  is optimal in the dual, then  $\mathbf{x}^T(\mathbf{c} - \mathbf{A}^T \mathbf{y}) = \mathbf{y}^T(\mathbf{b} - \mathbf{A} \mathbf{x}) = 0$ .

If  $\mathbf{x}$  is feasible in the primal,  $\mathbf{y}$  is feasible in the dual, and  $\mathbf{x}^T(\mathbf{c} - \mathbf{A}^T \mathbf{y}) = \mathbf{y}^T(\mathbf{b} - \mathbf{A} \mathbf{x}) = 0$ , then  $\mathbf{x}$  and  $\mathbf{y}$  are optimal for their respective problems.

# Relations between primal and dual optimal solutions

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primal (dual) problem	$\iff$	dual (primal) problem
unique and non-degenerate solution	$\iff$	unique and non-degenerate solution
unbounded solution	$\implies$	no feasible solutions
no feasible solutions	$\implies$	unbounded solution <b>or</b> no feasible solutions
degenerate solution	$\iff$	alternative solutions

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# Exercises on duality

## HOMEWORK!

- ▶ Formulate and solve graphically the dual of:

$$\begin{array}{ll} \text{minimize} & z = 6x_1 + 3x_2 + x_3 \\ \text{subject to} & 6x_1 - 3x_2 + x_3 \geq 2 \\ & 3x_1 + 4x_2 + x_3 \geq 5 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

- ▶ Then find the optimal primal solution
- ▶ Verify that the dual of the dual equals the primal