MVE165/MMG630, Applied Optimization Lecture 1 Introduction; course map; operations research; modelling optimization applications; graphic solution

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2012–03–13

Staff and homepage

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  - Mehdi Sharif Yazdi (Mathematical Sciences)
- **Course homepage & PingPong**
  - `www.math.chalmers.se/Math/Grundutb/CTH/mve165/1112`
  - `https://pingpong.gate.chalmers.se`
  - Details, information on assignmnents and computer exercises, deadlines, lecture notes, exercises etc
  - Will be updated with new information every course week

Course contents and organization

Contents

- Applications of optimization
- Mathematical modelling
- Theory – properties of models
- Solution techniques – algorithms
- Software solvers

Organization

- Lectures – mathematical optimization theory
- Computer exercises – get used to software solvers
- Guest lectures – applications of optimization
- Assignments – modelling, use solvers, written reports, opposition & oral presentation
- Assignment work should be done in groups of $\leq$ two persons
- Define your project groups on the PingPong page for MVE165
- GU students not having PingPong-entries: sign a list

Literature

- Main course book:
  - English version: Optimization (2010)
  - Swedish version: Optimeringslära (2008)

  by J. Lundgren, M. Rönnqvist, and P. Värbrand. Studentlitteratur.

- Exercise book:
  - English version: Optimization Exercises (2010)
  - Swedish version: Optimeringslära Övningsbok (2008)

  by M. Henningsson, J. Lundgren, M. Rönnqvist, and P. Värbrand. Studentlitteratur.

- Cremona/Studentlitteratur/Adlibris/...
- Hand-outs

Examination requirements

- Perform three project assignments in groups of two students
    - For Assignment 3 there will be three alternatives
- Written reports of three assignments
- A written opposition to another group's report of Assignment 2
- An oral presentation of Assignment 3
- Presence at one full oral presentation session
- To be able to receive a grade higher than 3 or G, the written reports and opposition as well as the oral presentation must be of high quality. Students aiming at grade 4, 5, or VG must also pass an oral exam

Overview of the lectures

- Linear optimization, modelling, theory, solution methods, sensitivity analysis
- Discrete optimization models, properties, and solution methods
- Optimization models that can be described as flows in networks, theory, and solution methods
- Multi-objective optimization
- Overview of non-linear optimization models, properties, and solution methods
- Mixes of the above

Optimization: “Do something as good as possible”

- **Something:** Which are the decision alternatives?
  - $x_{ij} = \begin{cases} 1 & \text{if customer } j \text{ is visited } \textit{directly} \text{ after customer } i \\ 0 & \text{else} \end{cases}$
  - $y_{ik} = \begin{cases} 1 & \text{if customer } i \text{ is visited by vehicle } k \\ 0 & \text{else} \end{cases}$
- **Possible:** What restrictions are there?
  - Each customer should be visited exactly once
  - Time windows, transport needs and capacity, pick up at one customer – deliver at another, different types of vehicles, ...
- **Good:** What is a relevant optimization criterion?
  - Minimize the total distance / travel time / emissions / waiting time / ...

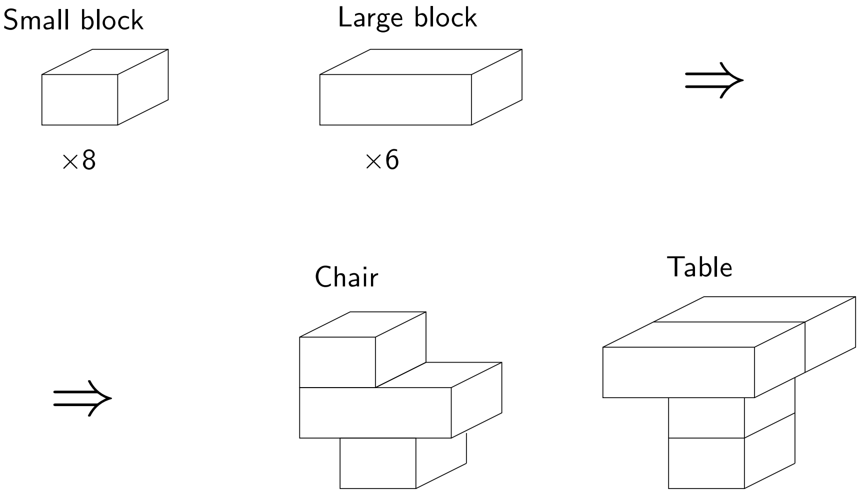
The classical travelling salesperson problem

- Variants of routing problems: e.g., refrigerated goods, transportation service for the disabled, school buses, ...

Examples of application areas

- **Logistics: production and transport**
  - Optimize routes for transports, snow removal, school buses, ...
  - Location of stores
  - Planning of wood cut and transports
  - Packing of containers
  - Production planning and scheduling
- **Energy**
  - Energy production planning
  - Investmen in techniques for energy production
  - Location of power plants and infrastructure
- **Finance**
  - Financial risk management
  - Portfolio optimization
  - Investment planning
- **Medicine**
  - Compute radiation directions/intensities at cancer treatment
  - Reconstruct images from x-ray measurements
  - Optimal timing for organ transplants

A manufacturing example: Produce tables and chairs from two types of blocks



A manufacturing example, continued

- A chair is assembled from one large and two small blocks
- A table is assembled from two blocks of each size
- Only 6 large and 8 small blocks are avaliable
- A table is sold at a revenue of 1600:-
- A chair is sold at a revenue of 1000:-
- Assume that all items produced can be sold and determine an optimal production plan

A mathematical optimization model

- **Something** – What decision alternatives? $\Rightarrow$ Variables

x_1 = number of tables produced and sold
x_2 = number of chairs produced and sold

- **Possible** – What restrictions? $\Rightarrow$ Constraints
  - Maximum supply of large blocks: 6

$$\boxed{2x_1 + x_2 \leq 6}$$

  - Maximum supply of small blocks: 8

$$\boxed{2x_1 + 2x_2 \leq 8}$$

  - Physical restrictions (also: x_1, x_2 integral)

$$\boxed{x_1, x_2 \geq 0}$$

- **Good** – Relevant optimization criterion? $\Rightarrow$ Objective function
  - Maximize the total revenue

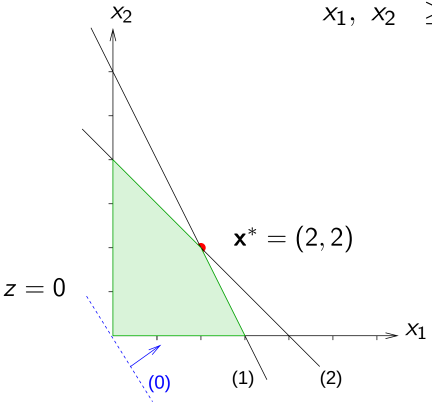
$$\boxed{1600x_1 + 1000x_2 \rightarrow \max}$$

Solve the model using LEGO!

- Start at no production: $x_1 = x_2 = 0$
  Use the "best marginal profit" to choose the item to produce
  - x_1 has the highest marginal profit (1600:-/table)
    $\Rightarrow$ produce as many tables as possible
  - At $x_1 = 3$: no more large blocks left
- The marginal value of x_2 is now 200:- since taking apart one table (-1600:-) yields two chairs ($+2000$:-) $\Rightarrow$ 400:-/2 chairs
  - Increase x_2 maximally $\Rightarrow$ decrease x_1
  - At $x_1 = x_2 = 2$: no more small blocks
- The marginal value of x_1 is negative (to build one more table one has to take apart two chairs $\Rightarrow$ -400:-)
  The marginal value of x_2 is -600:- (to build one more chair one table must be taken apart)
  $\Longrightarrow$ Optimal solution: $x_1 = x_2 = 2$

Geometric solution of the model

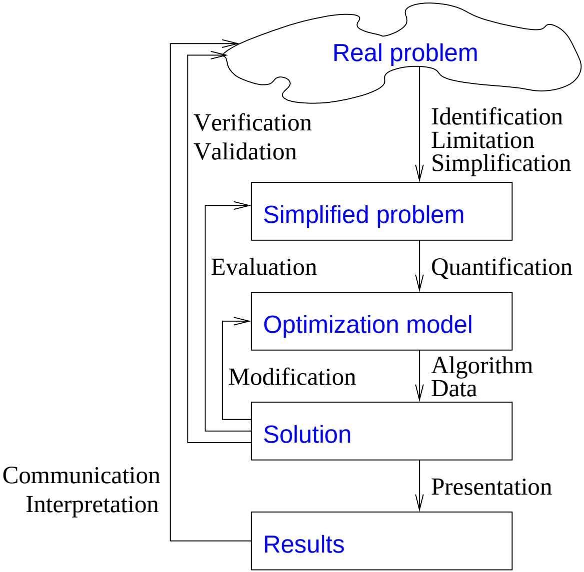
$$\begin{array}{llrcrcll}
\text{maximize} & z = & 1600x_1 & + & 1000x_2 & & & (0)\\
\text{subject to} & & 2x_1 & + & x_2 & \leq & 6 & (1)\\
 & & 2x_1 & + & 2x_2 & \leq & 8 & (2)\\
 & & & & x_1,\ x_2 & \geq & 0 &
\end{array}$$



Operations Research (OR) (Swedish: Operationsanalys)

- Scientific view on problem solving regarding complex systems
- *"OR means to prepare a foundation for rational decisions by utilizing systematic scientific methods and — where possible and meaningful — by utilizing quantitative models"*
- The problem is considered as a system of components which cooperate and influence each other
- The activities studied are described by models, used to
  - better understand the depicted system,
  - understand the consequences of different decisions, and
  - choose the "best" alternative due to some criterion.

The process of optimization

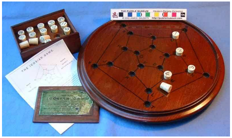


History of Operations Research

- During world war II decision problems became systematically treated: Operations Research
- After the war: use of operations research for civil operations
- The ideas spread to many countries
- Early operations research include inventory planning

A few moments in optimization history

- Euler (1735): Seven bridges of Köningsberg (graph theory)
- Gauss (1777–1855): 1st optimization technique, steepest descent
- W.R. Hamilton (1857): “icosian game”
  $\Rightarrow$ the travelling salesperson problem (Hamilton cycle)



- L.V. Kantorovich (1939): A linear model for optimization of plywood manufactoring and an algorithm for its solution
- George B. Dantzig (1947): Linear programming – the simplex algorithm (exponential time)
  - Program $\Leftrightarrow$ military training and logistics schedules
- Kantorovich and Koopmans (1975): Nobel Memorial Prize in Economic Sciences
- N. Karmarkar (1984) algorithm for linear programming (polynomial time)

Optimization modelling: A production–inventory example

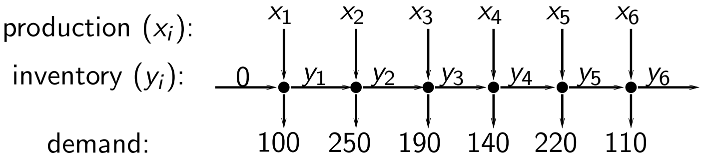
- Commission: Deliver windows over a six-month period
- Demand during the respective months: 100, 250, 190, 140, 220 & 110 units
- Production cost per unit (window): 50 €,45 €, 55 €, 48 €, 52 € & 50 €
- Store a manufactured window from one month to the next at 8 €
- Requirement: Meet the demand and minimize the costs
- Find an optimal production schedule

Define the decision variables

$$x_i \quad = \quad \text{number of units produced in month } i = 1, \ldots, 6$$

$$y_i \quad = \quad \text{units left in the inventory at the end of month } i = 1, \ldots, 6$$

- The "flow" of windows can be illustrated as:



Define the limitations/constraints

- Each month:

initial inventory + production − ending inventory = demand

$$\begin{array}{ccccccccc} 0 & + & x_1 & - & y_1 & = & 100 & & \\ y_1 & + & x_2 & - & y_2 & = & 250 & & \\ y_2 & + & x_3 & - & y_3 & = & 190 & & \\ y_3 & + & x_4 & - & y_4 & = & 140 & & \\ y_4 & + & x_5 & - & y_5 & = & 220 & & \\ y_5 & + & x_6 & - & y_6 & = & 110 & & \\ & & x_i & , & y_i & \geq & 0, & i = 1, \ldots, 6 \end{array}$$

Objective function: minimize the costs for production and storage

- Production cost (€):

$$50\ x_1 + 45\ x_2 + 55\ x_3 + 48\ x_4 + 52\ x_5 + 50\ x_6$$

- Inventory cost (€):

$$8\ (y_1 + y_2 + y_3 + y_4 + y_5 + y_6)$$

- Objective:

$$\begin{array}{ll} \text{minimize} & 50x_1 + 45x_2 + 55x_3 + 48x_4 + 52x_5 + 50x_6 \\ & +8(y_1 + y_2 + y_3 + y_4 + y_5 + y_6) \end{array}$$

A complete (general) optimization model

$$
\begin{array}{rrcll}
\text{minimize} & \displaystyle\sum_{i=1}^{6} c_i x_i + 8 \sum_{i=1}^{6} y_i, & & & \\
\text{subject to} & y_{i-1} + x_i - y_i & = & d_i, & i = 1, \ldots, 6, \\
& y_0 & = & 0, & \\
& x_i, y_i & \geq & 0, & i = 1, \ldots, 6,
\end{array}
$$

The vector of demand:

$$d = (d_i)_{i=1}^{6} = (100, 250, 190, 140, 220, 110)$$

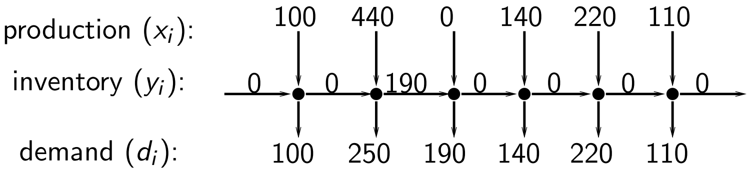
The vector of production costs:

$$c = (c_i)_{i=1}^{6} = (50, 45, 55, 48, 52, 50) \qquad .$$

An optimal solution—optimal production schedule

Optimal production each month:
$x = (x_i)_{i=1}^{6} = (100, 440, 0, 140, 220, 110)$

Optimal inventory each month:
$y = (y_i)_{i=0}^{6} = (0, 0, 190, 0, 0, 0, 0)$



The minimal total cost is 49980 €

Mathematical optimization models

$$\left[\begin{array}{lll} \text{minimize or maximize} & f(x_1,\ldots,x_n) & \\ \text{subject to} & g_i(x_1,\ldots,x_n) \left\{\begin{array}{c}\leq\\=\\\geq\end{array}\right\} b_i, & i=1,\ldots,m \end{array}\right]$$

- $x_1,\ldots,x_n$ are the decision variables
- f and $g_1,\ldots,g_m$ are given functions of the decision variables
- $b_1,\ldots,b_m$ are specified constant parameters
- The functions can be nonlinear, e.g. quadratic, exponential, logarithmic, non-analytic, ...
- In general, linear forms are more tractable than non-linear

Linear optimization models (programs)

- The production inventory model is a linear program (LP), i.e., all relations are described by linear forms
- A general linear program:

$$\left[\begin{array}{llllll} \text{min or max} & c_1x_1 + \ldots + c_nx_n & & & & \\ \text{subject to} & a_{i1}x_1 + \ldots + a_{in}x_n & \left\{\begin{array}{c} \leq \\ = \\ \geq \end{array}\right\} & b_i, & i = 1, \ldots, m \\ & \qquad\qquad\qquad x_j & \geq & 0, & j = 1, \ldots, n \end{array}\right]$$

- The non-negativity constraints on x_j, $j = 1, \ldots, n$ are not necessary, but usually assumed (reformulation always possible)

Discrete/integer/binary modelling

- A variable is called *discrete* if it can take only a countable set of values, e.g.,
  - Continuous variable: $x \in [0, 8] \iff 0 \leq x \leq 8$
  - Discrete variable: $x \in \{0, 4.4, 5.2, 8.0\}$
  - *Integer* variable: $x \in \{0, 1, 4, 5, 8\}$
- A *binary* variable can only take the values 0 or 1, i.e., all or nothing

  E.g., a wind-mill can produce electricity only if it is built
  - Let $y = 1$ if the mill is built, otherwise $y = 0$
  - Capacity of a mill: C
  - Production $x \leq Cy$ (also limited by wind force etc.)
- In general, models with only continuous variables are more tractable than models with integrality/discrete requirements on the variables, but exceptions exist! More about this later.

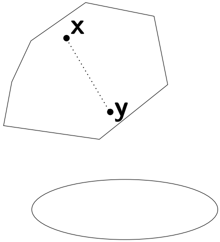
Convex sets

- A set S is convex if, for any elements $\mathbf{x}, \mathbf{y} \in S$ it holds that

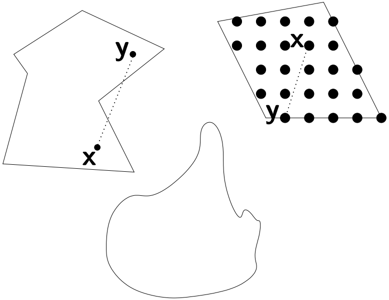
$$\alpha \mathbf{x} + (1 - \alpha)\mathbf{y} \in S \text{ for all } 0 \leq \alpha \leq 1$$

- Examples:

Convex sets



Non-convex sets



$\Rightarrow$ Intersections of linear (in)equalities $\Rightarrow$ convex sets

Convex and concave functions

- A function f is convex on the set S if, for any elements $\mathbf{x}, \mathbf{y} \in S$ it holds that

$$f(\alpha \mathbf{x} + (1-\alpha)\mathbf{y}) \leq \alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y}) \text{ for all } 0 \leq \alpha \leq 1$$

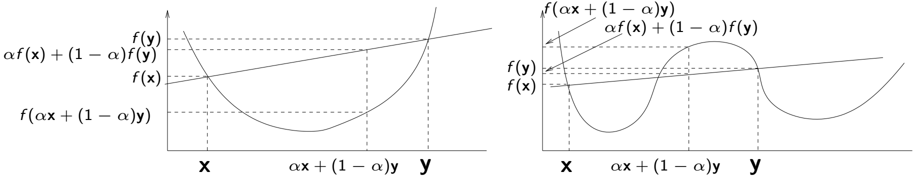
- A function f is concave on the set S if, for any elements $\mathbf{x}, \mathbf{y} \in S$ it holds that

$$f(\alpha \mathbf{x} + (1-\alpha)\mathbf{y}) \geq \alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y}) \text{ for all } 0 \leq \alpha \leq 1$$

$\Rightarrow$ Linear functions are convex (and concave)

Convex function

Non-convex function



Global solutions of convex and linear optimization problem

- [Theorem 11.3] Let $\mathbf{x}^*$ be a *local* minimizer of a *convex function* over a *convex set*. Then $\mathbf{x}^*$ is also a *global* minimizer.

$\Rightarrow$ Every *local optimum* of a linear optimization problem is a *global optimum*

- If a linear optimization problem has any optimal solutions, at least one optimal solution is at an *extreme point* of the feasible set

$\Rightarrow$ Search for optimal extreme point(s)

- Next lecture: Linear optimization problems and the simplex method