MVE165/MMG630, Applied Optimization
Lecture 2
Convexity; basic feasible solutions; the simplex method

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Mathematical optimization models

$$\left[\begin{array}{lll} \text{minimize} \ \textsf{or} \ \text{maximize} & f(x_1,\ldots,x_n) & \\ \textsf{subject to} & g_i(x_1,\ldots,x_n) \ \left\{\begin{array}{c}\leq\\=\\\geq\end{array}\right\} \ b_i, & i=1,\ldots,m \end{array}\right]$$

- $x_1,\ldots,x_n$ are the decision variables
- f and $g_1,\ldots,g_m$ are given functions of the decision variables
- $b_1,\ldots,b_m$ are specified constant parameters
- The functions can be nonlinear, e.g. quadratic, exponential, logarithmic, non-analytic, ...
- In general, linear forms are more tractable than non-linear

Linear optimization models (programs)

- The production inventory model is a linear program (LP), i.e., all relations are described by linear forms
- A general linear program:

$$\left[\begin{array}{lllll} \text{min or max} & c_1x_1 + \ldots + c_nx_n & & & \\ \text{subject to} & a_{i1}x_1 + \ldots + a_{in}x_n & \left\{\begin{array}{c} \leq \\ = \\ \geq \end{array}\right\} & b_i, & i = 1, \ldots, m \\ & x_j & \geq & 0, & j = 1, \ldots, n \end{array}\right]$$

- The non-negativity constraints on x_j, $j = 1, \ldots, n$ are not necessary, but usually assumed (reformulation always possible)

Discrete/integer/binary modelling

- A variable is called *discrete* if it can take only a countable set of values, e.g.,
  - Continuous variable: $x \in [0, 8] \iff 0 \leq x \leq 8$
  - Discrete variable: $x \in \{0, 4.4, 5.2, 8.0\}$
  - *Integer* variable: $x \in \{0, 1, 4, 5, 8\}$
- A *binary* variable can only take the values 0 or 1, i.e., all or nothing

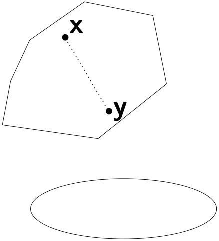
  E.g., a wind-mill can produce electricity only if it is built
  - Let $y = 1$ if the mill is built, otherwise $y = 0$
  - Capacity of a mill: C
  - Production $x \leq Cy$ (also limited by wind force etc.)
- In general, models with only continuous variables are more tractable than models with integrality/discrete requirements on the variables, but exceptions exist! More about this later.

Convex sets

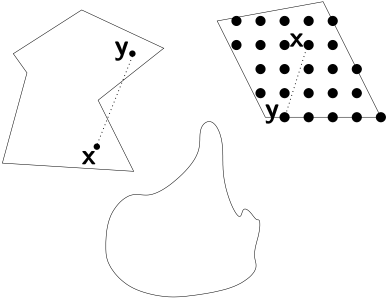
- A set S is convex if, for any elements $\mathbf{x}, \mathbf{y} \in S$ it holds that

$$\alpha \mathbf{x} + (1-\alpha)\mathbf{y} \in S \text{ for all } 0 \leq \alpha \leq 1$$

- Examples:



Convex sets



Non-convex sets

$\Rightarrow$ Intersections of linear (in)equalities $\Rightarrow$ convex sets

Convex and concave functions

- A function f is convex on the set S if, for any elements $\mathbf{x}, \mathbf{y} \in S$ it holds that

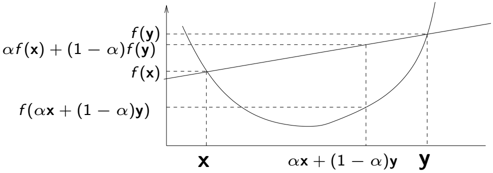
$$f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \leq \alpha f(\mathbf{x}) + (1-\alpha)f(\mathbf{y}) \text{ for all } 0 \leq \alpha \leq 1$$

- A function f is concave on the set S if, for any elements $\mathbf{x}, \mathbf{y} \in S$ it holds that

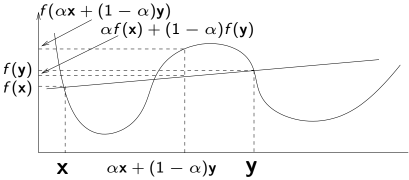
$$f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \geq \alpha f(\mathbf{x}) + (1-\alpha)f(\mathbf{y}) \text{ for all } 0 \leq \alpha \leq 1$$

$\Rightarrow$ Linear functions are convex (and concave)

Convex function



Non-convex function



Global solutions of convex and linear optimization problem

- [Theorem 11.3] Let $\mathbf{x}^*$ be a *local* minimizer of a *convex function* over a *convex set*. Then $\mathbf{x}^*$ is also a *global* minimizer.

⇒ Every local optimum of a linear optimization problem is a global optimum

- If a linear optimization problem has any optimal solutions, at least one optimal solution is at an extreme point of the feasible set

⇒ Search for optimal extreme point(s)

- Next lecture: Linear optimization problems and the simplex method

A general linear program – notation

$$\text{minimize} \;\; \text{or} \;\; \text{maximize} \quad c_1x_1 + \ldots + c_nx_n$$

$$\text{subject to} \quad a_{i1}x_1 + \ldots + a_{in}x_n \quad \left\{ \begin{array}{c} \leq \\ = \\ \geq \end{array} \right\} \quad b_i, \quad i = 1, \ldots, m$$

$$x_j \quad \left\{ \begin{array}{l} \leq 0 \\ \text{unrestricted in sign} \\ \geq 0 \end{array} \right\}, \quad j = 1, \ldots, n$$

- c_j, a_{ij}, and b_i are constant parameters for $i = 1, \ldots, m$ and $j = 1, \ldots, n$

The standard form and the simplex method for linear programs

- Every linear program can be reformulated such that:
  - all constraints are expressed as equalities with non-negative right hand sides
  - all variables are restricted to be non-negative
- Referred to as the *standard form*
- These requirements streamline the calculations of the *simplex method*
- Software solvers (e.g., Cplex, GLPK, Clp) can handle also inequality constraints and unrestricted variables – the reformulations are made automatically

The simplex method—reformulations

- The lego example:

$$\left[\begin{array}{rrl} 2x_1 & +x_2 \leq & 6 \\ 2x_1 & +2x_2 \leq & 8 \\ & x_1, x_2 \geq & 0 \end{array}\right] \Leftrightarrow \left[\begin{array}{rrrrl} 2x_1 & +x_2 & +s_1 & = & 6 \\ 2x_1 & +2x_2 & & +s_2 = & 8 \\ & & & x_1, x_2, s_1, s_2 \geq & 0 \end{array}\right]$$

- s_1 and s_2 are called *slack variables*—they "fill out" the (positive) distances between the left and right hand sides
- *Surplus variable* s_3 (a different example):

$$\left[\begin{array}{rrrcr} x_1 & + & x_2 & \geq & 800 \\ & & x_1, x_2 & \geq & 0 \end{array}\right] \Leftrightarrow \left[\begin{array}{rrrrrcr} x_1 & + & x_2 & - & s_3 & = & 800 \\ & & & & x_1, x_2, s_3 & \geq & 0 \end{array}\right]$$

The simplex method—reformulations, cont.

- Non-negative right hand side:

$$\begin{bmatrix} x_1 - x_2 & \leq -23 \\ x_1, x_2 & \geq 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} -x_1 + x_2 & \geq 23 \\ x_1, x_2 & \geq 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} -x_1 + x_2 - s_4 & = 23 \\ x_1, x_2, s_4 & \geq 0 \end{bmatrix}$$

- Sign-restricted (non-negative) variables:

$$\begin{bmatrix} x_1 + x_2 & \leq 10 \\ x_1 & \geq 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} x_1 + x_2^1 - x_2^2 & \leq 10 \\ x_1, x_2^1, x_2^2 & \geq 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} x_1 + x_2^1 - x_2^2 + s_5 & = 10 \\ x_1, x_2^1, x_2^2, s_5 & \geq 0 \end{bmatrix}$$

Basic feasible solutions

- Consider m equations of n variables, where $m \leq n$
- Set $n - m$ variables to zero and solve (if possible) the remaining $(m \times m)$ system of equations
- If the solution is *unique*, it is called a *basic* solution
- A basic solution corresponds to an *intersection* (feasible $(x \geq 0)$ or infeasible $(x \ngeq 0)$) of m hyperplanes in $\Re^m$
- Each *extreme point* of the feasible set is an intersection of m hyperplanes such that all variable values are ≥ 0
- Basic feasible solution $\Leftrightarrow$ extreme point of the feasible set

$$\begin{array}{ccc}
a_{11}x_1 + \ldots + a_{1n}x_n = b_1 & \qquad & x_1 \geq 0 \\
a_{21}x_1 + \ldots + a_{2n}x_n = b_2 & & x_2 \geq 0 \\
\ldots & & \ldots \\
a_{m1}x_1 + \ldots + a_{mn}x_n = b_m & & x_n \geq 0
\end{array}$$

Basic feasible solutions, example

- Constraints:

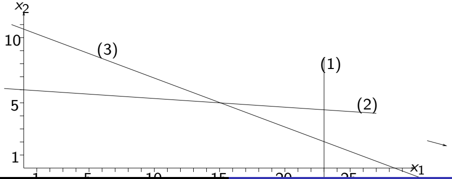
$$\begin{array}{rcrcrl} x_1 & & & \leq & 23 & (1) \\ 0.067x_1 & + & x_2 & \leq & 6 & (2) \\ 3x_1 & + & 8x_2 & \leq & 85 & (3) \\ & & x_1, x_2 & \geq & 0 & \end{array}$$

- Add slack variables:

$$\begin{array}{rrrrrll} x_1 & & +s_1 & & & = 23 & (1) \\ 0.067x_1 & +x_2 & & +s_2 & & = 6 & (2) \\ 3x_1 & +8x_2 & & & +s_3 & = 85 & (3) \\ & & & & x_1, x_2, s_1, s_2, s_3 & \geq 0 & \end{array}$$

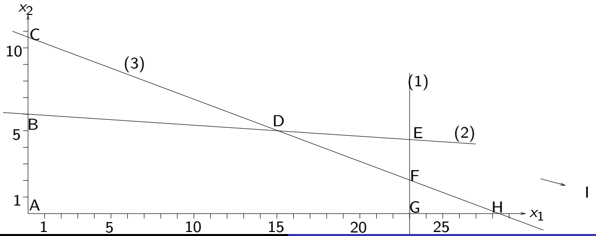
$m = 3$

$n = 5$

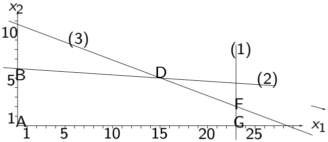


Basic and non-basic variables and solutions

basic variables	basic solution			non-basic variables $(0, 0)$	point	feasible?
s_1, s_2, s_3	23	6	85	x_1, x_2	A	yes
s_1, s_2, x_1	$-5\frac{1}{3}$	$4\frac{1}{9}$	$28\frac{1}{3}$	s_3, x_2	H	no
s_1, s_2, x_2	23	$-4\frac{5}{8}$	$10\frac{5}{8}$	x_1, s_3	C	no
s_1, x_1, s_3	-67	90	-185	s_2, x_2	I	no
s_1, x_2, s_3	23	6	37	s_2, x_1	B	yes
x_1, s_2, s_3	23	$4\frac{7}{15}$	16	s_1, x_2	G	yes
x_2, s_2, s_3	-	-	-	s_1, x_1	-	-
x_1, x_2, s_1	15	5	8	s_2, s_3	D	yes
x_1, x_2, s_2	23	2	$2\frac{7}{15}$	s_1, s_3	F	yes
x_1, x_2, s_3	23	$4\frac{7}{15}$	$-19\frac{11}{15}$	s_1, s_2	E	no



Basic feasible solutions correspond to solutions to the system of equations that fulfil non-negativity



$$\left[\begin{array}{rrrrrl} x_1 & & +s_1 & & & = 23 \\ 0.067x_1 & +x_2 & & +s_2 & & = 6 \\ 3x_1 & +8x_2 & & & +s_3 & = 85 \end{array}\right]$$

A: $x_1 = x_2 = 0 \Rightarrow \left[\begin{array}{llll} s_1 & & & = 23 \\ & s_2 & & = 6 \\ & & s_3 & = 85 \end{array}\right]$

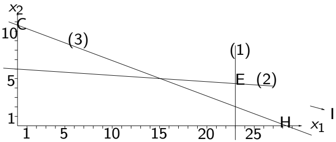
B: $x_1 = s_2 = 0 \Rightarrow \left[\begin{array}{rrrl} & s_1 & & = 23 \\ x_2 & & & = 6 \\ 8x_2 & & +s_3 & = 85 \end{array}\right]$

D: $s_3 = s_2 = 0 \Rightarrow \left[\begin{array}{rrrl} x_1 & & +s_1 & = 23 \\ 0.067x_1 & +x_2 & & = 6 \\ 3x_1 & +8x_2 & & = 85 \end{array}\right]$

F: $s_3 = s_1 = 0 \Rightarrow \left[\begin{array}{rrrl} x_1 & & & = 23 \\ 0.067x_1 & +x_2 & +s_2 & = 6 \\ 3x_1 & +8x_2 & & = 85 \end{array}\right]$

G: $x_2 = s_1 = 0 \Rightarrow \left[\begin{array}{rrrl} x_1 & & & = 23 \\ 0.067x_1 & +s_2 & & = 6 \\ 3x_1 & & +s_3 & = 85 \end{array}\right]$

Basic infeasible solutions corresp. to solutions to the system of equations with one or more variables < 0



$$\left[\begin{array}{rrrrrl} x_1 & & +s_1 & & & = 23 \\ 0.067x_1 & +x_2 & & +s_2 & & = 6 \\ 3x_1 & +8x_2 & & & +s_3 & = 85 \end{array}\right]$$

H: $x_2 = s_3 = 0 \Rightarrow \left[\begin{array}{rrrl} x_1 & +s_1 & & = 23 \\ 0.067x_1 & & +s_2 & = 6 \\ 3x_1 & & & = 85 \end{array}\right]$

C: $x_1 = s_3 = 0 \Rightarrow \left[\begin{array}{rrrl} & s_1 & & = 23 \\ x_2 & & +s_2 & = 6 \\ 8x_2 & & & = 85 \end{array}\right]$

I: $s_2 = x_2 = 0 \Rightarrow \left[\begin{array}{rrrl} x_1 & +s_1 & & = 23 \\ 0.067x_1 & & & = 6 \\ 3x_1 & & +s_3 & = 85 \end{array}\right]$

-: $s_1 = x_1 = 0 \Rightarrow \left[\begin{array}{rrrrl} & & & 0 & = 23 \\ x_2 & +s_2 & & & = 6 \\ 8x_2 & & +s_3 & & = 85 \end{array}\right]$

E: $s_1 = s_2 = 0 \Rightarrow \left[\begin{array}{rrrl} x_1 & & & = 23 \\ 0.067x_1 & +x_2 & & = 6 \\ 3x_1 & +8x_2 & +s_3 & = 85 \end{array}\right]$

Basic feasible solutions and the simplex method

- Express the m *basic* variables in terms of the $n - m$ *non-basic* variables
- Example: Start at $x_1 = x_2 = 0 \Rightarrow s_1, s_2, s_3$ are *basic*

$$\left[\begin{array}{cccccc} x_1 & & +s_1 & & & = 23 \\ \frac{1}{15}x_1 & +x_2 & & +s_2 & & = 6 \\ 3x_1 & +8x_2 & & & +s_3 & = 85 \end{array}\right]$$

- Express s_1, s_2, and s_3 in terms of x_1 and x_2 (*non-basic*):

$$\left[\begin{array}{rrrr} s_1 = & 23 & -x_1 & \\ s_2 = & 6 & -\frac{1}{15}x_1 & -x_2 \\ s_3 = & 85 & -3x_1 & -8x_2 \end{array}\right]$$

- We wish to maximize the objective function $2x_1 + 3x_2$
- Express the objective in terms of the *non-basic* variables:

$$\text{(maximize)} \quad z = 2x_1 + 3x_2 \quad \Leftrightarrow \quad z - 2x_1 - 3x_2 = 0$$

Basic feasible solutions and the simplex method

- The *first basic solution* can be represented as

$$\begin{array}{rrrrrrr|l}
-z & +2x_1 & +3x_2 & & & & = 0 & (0) \\
 & x_1 & & +s_1 & & & = 23 & (1) \\
 & \frac{1}{15}x_1 & +x_2 & & +s_2 & & = 6 & (2) \\
 & 3x_1 & +8x_2 & & & +s_3 & = 85 & (3)
\end{array}$$

- Marginal values for increasing the non-basic variables x_1 and x_2 from zero: 2 and 3, resp.

⇒ Choose x_2 — let x_2 *enter the basis* DRAW GRAPH!!

- One basic variable (s_1, s_2, or s_3) must *leave the basis*. Which?
- The value of x_2 can increase until some basic variable reaches the value 0:

$$\left.\begin{array}{ll}
(2): s_2 = 6 - x_2 \geq 0 & \Rightarrow x_2 \leq 6 \\
(3): s_3 = 85 - 8x_2 \geq 0 & \Rightarrow x_2 \leq 10\frac{5}{8}
\end{array}\right\} \Rightarrow \begin{array}{c} s_2 = 0 \text{ when} \\ x_2 = 6 \\ (\text{and } s_3 = 37) \end{array}$$

- s_2 will leave the basis

Change basis through row operations

- Eliminate s_2 from the basis, let x_2 enter the basis using row operations:

$$\begin{array}{llllll|cr|l}
\hline
-z & +2x_1 & +3x_2 & & & & = & 0 & (0) \\
 & x_1 & & +s_1 & & & = & 23 & (1) \\
 & \frac{1}{15}x_1 & +x_2 & & +s_2 & & = & 6 & (2) \\
 & 3x_1 & +8x_2 & & & +s_3 & = & 85 & (3) \\
\hline
-z & +\frac{9}{5}x_1 & & & -3s_2 & & = & -18 & (0)\ -3\cdot(2) \\
 & x_1 & & +s_1 & & & = & 23 & (1)-0\cdot(2) \\
 & \frac{1}{15}x_1 & +x_2 & & +s_2 & & = & 6 & (2) \\
 & \frac{37}{15}x_1 & & & -8s_2 & +s_3 & = & 37 & (3)-8\cdot(2) \\
\hline
\end{array}$$

- Corresponding basic solution: $s_1 = 23$, $x_2 = 6$, $s_3 = 37$.
- Nonbasic variables: $x_1 = s_2 = 0$
- The marginal value of x_1 is $\frac{9}{5} > 0$. Let x_1 enter the basis
- Which one should leave? s_1, x_2, or s_3?

Change basis ...

$$\begin{array}{rrrrrr|cr|l}
-z & +\frac{9}{5}x_1 & & & -3s_2 & & = & -18 & (0)\\
 & x_1 & & +s_1 & & & = & 23 & (1)\\
 & \frac{1}{15}x_1 & +x_2 & & +s_2 & & = & 6 & (2)\\
 & \boldsymbol{\frac{37}{15}x_1} & & & -8s_2 & +s_3 & = & 37 & (3)
\end{array}$$

- The value of x_1 can increase until some basic variable reaches the value 0:

$$\left.\begin{array}{ll}
(1) : s_1 = 23 - x_1 \geq 0 & \Rightarrow x_1 \leq 23\\
(2) : x_2 = 6 - \frac{1}{15}x_1 \geq 0 & \Rightarrow x_1 \leq 90\\
(3) : s_3 = 37 - \frac{37}{15}x_1 \geq 0 & \Rightarrow x_1 \leq 15
\end{array}\right\} \Rightarrow \begin{array}{c} s_3 = 0 \text{ when}\\ x_1 = 15 \end{array}$$

- x_1 enters the basis and s_3 leaves the basis
- Perform row operations:

$$\begin{array}{rrrrrr|cr|l}
-z & & & & +2.84s_2 & -0.73s_3 & = & -45 & (0)-(3)\cdot\frac{15}{37}\cdot\frac{9}{5}\\
 & & & s_1 & +3.24s_2 & -0.41s_3 & = & 8 & (1)-(3)\cdot\frac{15}{37}\\
 & & x_2 & & +1.22s_2 & -0.03s_3 & = & 5 & (2)-(3)\cdot\frac{15}{37}\cdot\frac{1}{15}\\
 & x_1 & & & -3.24s_2 & +0.41s_3 & = & 15 & (3)\cdot\frac{15}{37}
\end{array}$$

Change basis ...

$$\begin{array}{cccccc|cr|c}
-z & & & & +2.84s_2 & -0.73s_3 & = & -45 & (0)\\
 & & & s_1 & +\mathbf{3.24s_2} & -0.41s_3 & = & 8 & (1)\\
 & & x_2 & & +1.22s_2 & -0.03s_3 & = & 5 & (2)\\
 & x_1 & & & -3.24s_2 & +0.41s_3 & = & 15 & (3)
\end{array}$$

- Let s_2 enter the basis (marginal value > 0)
- The value of s_2 can increase until some basic variable $= 0$:

$$\left.\begin{array}{ll}
(1) : s_1 = 8 - 3.24s_2 \geq 0 & \Rightarrow s_2 \leq 2.47\\
(2) : x_2 = 5 - 1.22s_2 \geq 0 & \Rightarrow s_2 \leq 4.10\\
(3) : x_1 = 15 + 3.24s_2 \geq 0 & \Rightarrow s_2 \geq -4.63
\end{array}\right\} \Rightarrow \begin{array}{c} s_1 = 0 \text{ when}\\ s_2 = 2.47 \end{array}$$

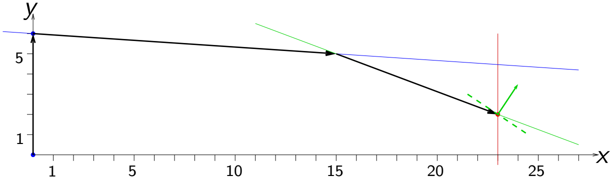
- s_2 enters the basis and s_1 will leave the basis
- Perform row operations:

$$\begin{array}{cccccc|cr|l}
-z & & & -0.87s_1 & & -0.37s_3 & = & -52 & (0)-(1)\cdot\frac{2.84}{3.24}\\
 & & & 0.31s_1 & +s_2 & -0.12s_3 & = & 2.47 & (1)\cdot\frac{1}{3.24}\\
 & & x_2 & -0.37s_1 & & +0.12s_3 & = & 2 & (2)-(1)\cdot\frac{1.22}{3.24}\\
 & x_1 & & +s_1 & & & = & 23 & (3)+(1)
\end{array}$$

Optimal basic solution

$$\begin{array}{cccccc|cr}
-z & & & -0.87s_1 & & -0.37s_3 & = & -52 \\
 & & & 0.31s_1 & +s_2 & -0.12s_3 & = & 2.47 \\
 & & x_2 & -0.37s_1 & & +0.12s_3 & = & 2 \\
 & x_1 & & +s_1 & & & = & 23
\end{array}$$

- No marginal value is positive. No improvement can be made
- The optimal basis is given by $s_2 = 2.47$, $x_2 = 2$, and $x_1 = 23$
- Non-basic variables: $s_1 = s_3 = 0$
- Optimal value: $z = 52$



Summary of the solution course

basis	$-z$	x_1	x_2	s_1	s_2	s_3	RHS
$-z$	1	2	3	0	0	0	0
s_1	0	1	0	1	0	0	23
s_2	0	0.067	**1**	0	1	0	6
s_3	0	3	8	0	0	1	85
$-z$	1	1.80	0	0	-3	0	-18
s_1	0	1	0	1	0	0	23
x_2	0	0.07	1	0	1	0	6
s_3	0	**2.47**	0	0	-8	1	37
$-z$	1	0	0	0	2.84	-0.73	-45
s_1	0	0	0	1	**3.24**	-0.41	8
x_2	0	0	1	0	1.22	-0.03	5
x_1	0	1	0	0	-3.24	0.41	15
$-z$	1	0	0	-0.87	0	-0.37	-52
s_2	0	0	0	0.31	1	-0.12	2.47
x_2	0	0	1	-0.37	0	0.12	2
x_1	0	1	0	1	0	0	23

Solve the lego problem using the simplex method!

$$\begin{array}{lrcrcrcr} \text{maximize} & z & = & 1600x_1 & + & 1000x_2 & & \\ \text{subject to} & & & 2x_1 & + & x_2 & \leq & 6 \\ & & & 2x_1 & + & 2x_2 & \leq & 8 \\ & & & & & x_1,\ x_2 & \geq & 0 \end{array}$$

HOMEWORK!!