MVE165/MMG630, Applied Optimization
Lecture 3
The simplex method; degeneracy; unbounded solutions; infeasibility; starting solutions; duality; interpretation

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Summary of the simplex method

- **Optimality condition**: The *entering* variable in a maximization (minimization) problem should have the largest positive (negative) marginal value (reduced cost).

  The entering variable *determines a direction* in which the objective value increases (decreases).

  If all *reduced costs are negative* (positive), the current basis is *optimal*.

- **Feasibility condition**: The *leaving* variable is the one with smallest nonnegative quotient.

  Corresponds to the constraint that is “reached first”

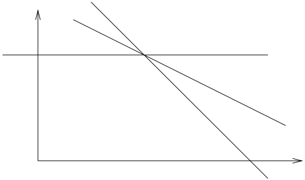
Simplex search for linear (minimization) programs (Ch. 4.6)

1. **Initialization:** Choose any feasible basis, construct the corresponding basic solution $\mathbf{x}^0$, let $t = 0$
2. **Step direction:** Select a variable to enter the basis using the optimality condition (negative marginal value). Stop if no entering variable exists
3. **Step length:** Select a leaving variable using the feasibility condition (smallest non-negative quotient)
4. **New iterate:** Compute the new basic solution $\mathbf{x}^{t+1}$ by performing matrix operations.
5. Let $t := t + 1$ and repeat from 2

Degeneracy (Ch. 4.10)

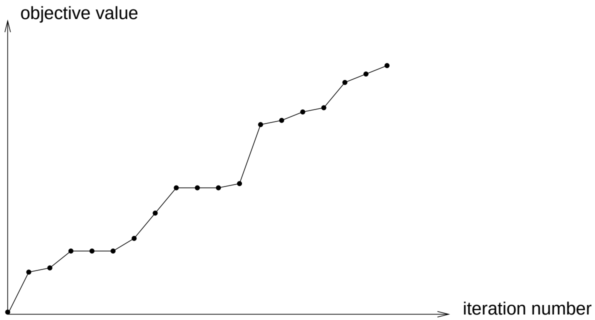
- If the smallest nonnegative quotient is zero, the value of a basic variable will become zero in the next iteration
- The solution is *degenerate*
- The objective value will *not* improve in this iteration
- Risk: *cycling* around (non-optimal) bases
- Reason: a *redundant* constraint "touches" the feasible set
- Example:

$$
\begin{array}{rcrcl}
x_1 & + & x_2 & \leq & 6 \\
 & & x_2 & \leq & 3 \\
x_1 & + & 2x_2 & \leq & 9 \\
 & & x_1,\ x_2 & \geq & 0
\end{array}
$$



Degeneracy

- Typical objective function progress of the simplex method



- Computation rules to prevent from infinite cycling: careful choices of leaving and entering variables
- In modern software: perturb the right hand side $(b_i + \Delta b_i)$, solve, reduce the perturbation and resolve starting from the current basis. Repeat until $\Delta b_i = 0$.

Unbounded solutions (Ch. 4.4, 4.6)

- If all quotients are *negative*, the value of the variable entering the basis may increase *infinitely*
- The feasible set is *unbounded*
- In a real application this would probably be due to some incorrect assumption
- Example:

$$\begin{array}{rrrrl} \text{minimize} & z = & -x_1 & -2x_2 & \\ \text{subject to} & & -x_1 & +x_2 & \leq 2 \\ & & -2x_1 & +x_2 & \leq 1 \\ & & & x_1,\ x_2 & \geq 0 \end{array}$$

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Unbounded solutions (Ch. 4.4, 4.6)

- A feasible basis is given by $x_1 = 1$, $x_2 = 3$, with corresponding tableau:

  Homework: Find this basis using the simplex method.

basis	$-z$	x_1	x_2	s_1	s_2	RHS
$-z$	1	0	0	5	-3	7
x_1	0	1	0	1	-1	1
x_2	0	0	1	2	-1	3

- Entering variable is s_2
- Row 1: $x_1 = 1 + s_2 \geq 0 \Rightarrow s_2 \geq -1$
- Row 2: $x_2 = 3 + s_2 \geq 0 \Rightarrow s_2 \geq -3$
- No leaving variable can be found, since no constraint will prevent s_2 from increasing infinitely

Starting solution—finding an initial basis (Ch. 4.9)

- Example:

$$
\begin{array}{lrrrl}
\text{minimize} & z = & 2x_1 & +3x_2 & \\
\text{subject to} & & 3x_1 & +2x_2 & = 14 \\
 & & 2x_1 & -4x_2 & \geq 2 \\
 & & 4x_1 & +3x_2 & \leq 19 \\
 & & & x_1, x_2 & \geq 0
\end{array}
$$

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- Add slack and surplus variables

$$
\begin{array}{lrrrrrl}
\text{minimize} & z = & 2x_1 & +3x_2 & & & \\
\text{subject to} & & 3x_1 & +2x_2 & & & = 14 \\
 & & 2x_1 & -4x_2 & -s_1 & & = 2 \\
 & & 4x_1 & +3x_2 & & +s_2 & = 19 \\
 & & & & & x_1, x_2, s_1, s_2 & \geq 0
\end{array}
$$

- How finding an initial basis? Only s_2 is obvious!

Artificial variables

- Add artificial variables a_1 and a_2 to the first and second constraints, respectively
- Solve an artificial problem: minimize $a_1 + a_2$

$$
\begin{array}{llrrrrrrcl}
\text{minimize} & w = & & & & & a_1 & +a_2 & & \\
\text{subject to} & & 3x_1 & +2x_2 & & & +a_1 & & = & 14 \\
& & 2x_1 & -4x_2 & -s_1 & & & +a_2 & = & 2 \\
& & 4x_1 & +3x_2 & & +s_2 & & & = & 19 \\
& & & & & & & x_1, x_2, s_1, s_2, a_1, a_2 & \geq & 0
\end{array}
$$

- The "phase one" problem
- An initial basis is given by $a_1 = 14$, $a_2 = 2$, and $s_2 = 19$:

basis	$-w$	x_1	x_2	s_1	s_2	a_1	a_2	RHS
$-w$	1	-5	2	1	0	0	0	-16
a_1	0	3	2	0	0	1	0	14
a_2	0	2	-4	-1	0	0	1	2
s_2	0	4	3	0	1	0	0	19

Find an initial solution using artificial variables

- x_1 enters $\Rightarrow$ a_2 leaves (then $x_2 \Rightarrow s_2$, then $s_1 \Rightarrow a_1$)

basis	$-w$	x_1	x_2	s_1	s_2	a_1	a_2	RHS
$-w$	1	-5	2	1	0	0	0	-16
a_1	0	3	2	0	0	1	0	14
a_2	0	2	-4	-1	0	0	1	2
s_2	0	4	3	0	1	0	0	19
$-w$	1	0	-8	-1.5	0	0		-11
a_1	0	0	8	1.5	0	1		11
x_1	0	1	-2	-0.5	0	0		1
s_2	0	0	11	2	1	0		15
$-w$	1	0	0	-0.045	0.727	0		-0.091
a_1	0	0	0	0.045	-0.727	1		0.091
x_1	0	1	0	-0.136	0.182	0		3.727
x_2	0	0	1	0.182	0.091	0		1.364
$-w$	1	0	0	0	0			0
s_1	0	0	0	1	-16			2
x_1	0	1	0	0	-2			4
x_2	0	0	1	0	3			1

- A feasible basis is given by $x_1 = 4$, $x_2 = 1$, and $s_1 = 2$

Infeasible linear programs (Ch. 4.9)

- If the solution to the "phase one" problem has optimal value $= 0$, a feasible basis has been found

$\Rightarrow$ Start optimizing the original objective function z from this basis (*homework*)

- If the solution to the "phase one" problem has optimal value $w > 0$, no feasible solutions exist
- What would this mean in a real application?
- Alternative: Big-M method: Add the artificial variables to the original objective—with a large coefficient
  Example:

$$\text{minimize} \quad z = 2x_1 + 3x_2$$

$$\Rightarrow \qquad \text{minimize} \quad z_a = 2x_1 + 3x_2 + Ma_1 + Ma_2$$

Alternative optimal solutions (Ch. 4.6)

- Example:

$$\begin{array}{rrrrl} \text{maximize} & z = & 2x_1 & +4x_2 & \\ \text{subject to} & & x_1 & +2x_2 & \leq 5 \\ & & x_1 & +x_2 & \leq 4 \\ & & & x_1, x_2 & \geq 0 \end{array}$$

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- The extreme points $(0, \frac{5}{2})$ and $(3, 1)$ have the same optimal value $z = 10$
- All solutions that are positive linear (convex) combinations of these are optimal:

$$(x_1, x_2) = \alpha \cdot (0, \frac{5}{2}) + (1 - \alpha) \cdot (3, 1), \quad 0 \leq \alpha \leq 1$$

A general linear program in standard form

- A linear program with n non-negative variables, m equality constraints ($m < n$), and non-negative right hand sides:

$$\begin{aligned} \text{maximize} \quad & z = \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1, \dots, m, \\ & x_j \geq 0, \quad j = 1, \dots, n. \end{aligned}$$

- On matrix form it is written as:

$$\begin{aligned} \text{maximize} \quad & z = \mathbf{c}^{\mathrm{T}} \mathbf{x}, \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}^n, \end{aligned}$$

where $\mathbf{x} \in \Re^n$, $\mathbf{A} \in \Re^{m \times n}$, $\mathbf{b} \in \Re^m_+$ ($\mathbf{b} \geq \mathbf{0}^m$), and $\mathbf{c} \in \Re^n$.

General derivation of the simplex method (Ch. 4.8)

- B = set of basic variables, N = set of non-basic variables

$\Rightarrow$ $|B| = m$ and $|N| = n - m$

- Partition matrix/vectors: $\mathbf{A} = (\mathbf{B}, \mathbf{N})$, $\mathbf{x} = (\mathbf{x}_B, \mathbf{x}_N)$, $\mathbf{c} = (\mathbf{c}_B, \mathbf{c}_N)$
- The matrix $\mathbf{B}$ ($\mathbf{N}$) contains the columns of $\mathbf{A}$ corresponding to the index set B (N) — Analogously for $\mathbf{x}$ and $\mathbf{c}$
- Rewrite the linear program:

$$\begin{bmatrix} \text{maximize } z = \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to } \mathbf{A}\mathbf{x} = \mathbf{b}, \\ \mathbf{x} \geq \mathbf{0}^n \end{bmatrix} = \begin{bmatrix} \text{maximize } z = \mathbf{c}_B^{\mathrm{T}}\mathbf{x}_B + \mathbf{c}_N^{\mathrm{T}}\mathbf{x}_N \\ \text{subject to } \mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_N = \mathbf{b}, \\ \mathbf{x}_B \geq \mathbf{0}^m, \ \mathbf{x}_N \geq \mathbf{0}^{n-m} \end{bmatrix}$$

- Substitute: $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N \implies$

$$\begin{aligned} \text{maximize} \quad & z = \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{b} + [\mathbf{c}_N^{\mathrm{T}} - \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{N}]\mathbf{x}_N \\ \text{subject to} \quad & \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N \geq \mathbf{0}^m, \\ & \mathbf{x}_N \geq \mathbf{0}^{n-m} \end{aligned}$$

Optimality and feasibility

- **Optimality condition** (for maximization)

  The basis B is optimal if $\mathbf{c}_N^{\mathrm{T}} - \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{N} \leq \mathbf{0}^{n-m}$
  (marginal values = reduced costs ≤ 0)

  If not, choose as entering variable $j \in N$ the one with the largest value of the reduced cost $c_j - \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{A}_j$

- **Feasibility condition**

  For all $i \in B$ it holds that $x_i = (\mathbf{B}^{-1}\mathbf{b})_i - (\mathbf{B}^{-1}\mathbf{A}_j)_i x_j$

  Choose the leaving variable $i^* \in B$ according to

$$i^* = \arg\min_{i \in B} \left\{ \frac{(\mathbf{B}^{-1}\mathbf{b})_i}{(\mathbf{B}^{-1}\mathbf{A}_j)_i} \,\middle|\, (\mathbf{B}^{-1}\mathbf{A}_j)_i > 0 \right\}$$

In the simplex tableau, we have

basis	$-z$	$\mathbf{x}_B$	$\mathbf{x}_N$	$\mathbf{s}$	RHS
$-z$	1	$\mathbf{0}$	$\mathbf{c}_N^{\mathrm{T}} - \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{N}$	$-\mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}$	$-\mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{b}$
$\mathbf{x}_B$	$\mathbf{0}$	$\mathbf{I}$	$\mathbf{B}^{-1}\mathbf{N}$	$\mathbf{B}^{-1}$	$\mathbf{B}^{-1}\mathbf{b}$

- $\mathbf{s}$ denotes possible slack variables (columns for $\mathbf{s}$ are copies of certain columns for $(\mathbf{x}_B, \mathbf{x}_N)$)
- The computations performed by the simplex algorithm involve matrix inversions and updates of these
- A non-basic (basic) variable enters (leaves) the basis $\Rightarrow$ one column, $\mathbf{A}_j$, of $\mathbf{B}$ is replaced by another, $\mathbf{A}_k$
- Row operations $\Leftrightarrow$ Updates of $\mathbf{B}^{-1}$ (and $\mathbf{B}^{-1}\mathbf{N}$, $\mathbf{B}^{-1}\mathbf{b}$, and $\mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}$)

$\Rightarrow$ Efficient numerical computations are crucial for the performance of the simplex algorithm

"Intuitive" derivation of duality (Ch. 6.1)

- A linear program with optimal value z^*

$$
\begin{array}{llrrlll}
\text{maximize} & z := & 20x_1 & +18x_2 & & & \text{weights} \\
\text{subject to} & & 7x_1 & +10x_2 & \leq 3600 & (1) & v_1 \\
& & 16x_1 & +12x_2 & \leq 5400 & (2) & v_2 \\
& & & x_1, x_2 & \geq 0 & &
\end{array}
$$

- How large can z^* be?
- Compute upper estimates of z^*, e.g.
  - Multiply (1) by 3 $\Rightarrow 21x_1 + 30x_2 \leq 10800 \Rightarrow z^* \leq 10800$
  - Multiply (2) by 1.5 $\Rightarrow 24x_1 + 18x_2 \leq 8100 \Rightarrow z^* \leq 8100$
  - Combine: $0.6 \times (1) + 1 \times (2) \Rightarrow 20.2x_1 + 18x_2 \leq 7560 \Rightarrow z^* \leq 7560$
- Do better than guess—compute optimal weights!
- Value of estimate: $w = 3600v_1 + 5400v_2 \rightarrow \min$
- Constraints on weights: $\left[\begin{array}{rl} 7v_1 + 16v_2 & \geq 20 \\ 10v_1 + 12v_2 & \geq 18 \\ v_1, v_2 & \geq 0 \end{array}\right]$

The best (lowest) possible upper estimate of z^*

$$
\begin{array}{rrcrl}
\text{minimize} & w := 3600v_1 & + & 5400v_2 & \\
\text{subject to} & 7v_1 & + & 16v_2 & \geq 20 \\
 & 10v_1 & + & 12v_2 & \geq 18 \\
 & & & v_1, v_2 & \geq 0
\end{array}
$$

- A linear program!
- It is called the **dual** of the original linear program

The lego model – the market problem

- Consider the lego problem

$$\begin{array}{lrcrcrcl} \text{maximize} & z & = & 1600x_1 & + & 1000x_2 & & \\ \text{subject to} & & & 2x_1 & + & x_2 & \leq & 6 \\ & & & 2x_1 & + & 2x_2 & \leq & 8 \\ & & & & & x_1, \ x_2 & \geq & 0 \end{array}$$

- Option: Sell bricks instead of making furniture
- $v_1(v_2)$ = price of a large (small) brick
- Market wish to minimize payment: *minimize* $6v_1 + 8v_2$
- I sell if prices are high enough:
  - $2v_1 + 2v_2 \geq 1600$ – otherwise better to make tables
  - $v_1 + 2v_2 \geq 1000$ – otherwise better to make chairs
  - $v_1, v_2 \geq 0$ – prices are naturally non-negative

Linear programming duality

- To each primal linear program corresponds a dual linear program

$$\begin{aligned} \text{[Primal]} \quad & \text{minimize} & z &= \mathbf{c}^{\mathrm{T}}\mathbf{x}, \\ & \text{subject to} & \mathbf{A}\mathbf{x} &= \mathbf{b}, \\ & & \mathbf{x} &\geq \mathbf{0}^n, \end{aligned}$$

$$\begin{aligned} \text{[Dual]} \quad & \text{maximize} & w &= \mathbf{b}^{\mathrm{T}}\mathbf{y}, \\ & \text{subject to} & \mathbf{A}^{\mathrm{T}}\mathbf{y} &\leq \mathbf{c}. \end{aligned}$$

- On component form:

$$\begin{array}{llllll} \text{[Primal]} & \text{minimize} & z = \sum_{j=1}^{n} c_j x_j & & & \\ & \text{subject to} & \sum_{j=1}^{n} a_{ij} x_j & = & b_i, & i = 1, \ldots, m, \\ & & x_j & \geq & 0, & j = 1, \ldots, n, \\ \hline \text{[Dual]} & \text{maximize} & w = \sum_{j=1}^{n} b_i y_i & & & \\ & \text{subject to} & \sum_{i=1}^{m} a_{ij} y_i & \leq & c_j, & j = 1, \ldots, n. \end{array}$$

An example

- A primal linear program

$$\begin{array}{rrrrl} \text{minimize} & z = & 2x_1 & +3x_2 & \\ \text{subject to} & & 3x_1 & +2x_2 & = 14 \\ & & 2x_1 & -4x_2 & \geq 2 \\ & & 4x_1 & +3x_2 & \leq 19 \\ & & & x_1, x_2 & \geq 0 \end{array}$$

- The corresponding dual linear program

$$\begin{array}{rrrrrl} \text{maximize} & w = & 14y_1 & +2y_2 & +19y_3 & \\ \text{subject to} & & 3y_1 & +2y_2 & +4y_3 & \leq 2 \\ & & 2y_1 & -4y_2 & +3y_3 & \leq 3 \\ & & y_1 & & & \text{free}, \\ & & & y_2 & & \geq 0, \\ & & & & y_3 & \leq 0 \end{array}$$

Rules for constructing the dual program (Ch. 6.2)

maximization	$\Leftrightarrow$	minimization
dual program	$\Leftrightarrow$	primal program
primal program	$\Leftrightarrow$	dual program
constraints		*variables*
$\geq$	$\Leftrightarrow$	≤ 0
$\leq$	$\Leftrightarrow$	≥ 0
$=$	$\Leftrightarrow$	free
variables		*constraints*
≥ 0	$\Leftrightarrow$	$\geq$
≤ 0	$\Leftrightarrow$	$\leq$
free	$\Leftrightarrow$	$=$

The dual of the dual of any linear program equals the primal

Duality properties (Ch. 6.3)

- **Weak duality**: Let $\mathbf{x}$ be a feasible point in the primal (minimization) and $\mathbf{y}$ be a feasible point in the dual (maximization). Then,

$$z = \mathbf{c}^{\mathrm{T}}\mathbf{x} \geq \mathbf{b}^{\mathrm{T}}\mathbf{y} = w$$

- **Strong duality**: In a pair of primal and dual linear programs, if one of them has an optimal solution, so does the other, and their optimal values are equal.
- **Complementary slackness**: If $\mathbf{x}$ is optimal in the primal and $\mathbf{y}$ is optimal in the dual, then $\mathbf{x}^{\mathrm{T}}(\mathbf{c} - \mathbf{A}^{\mathrm{T}}\mathbf{y}) = \mathbf{y}^{\mathrm{T}}(\mathbf{b} - \mathbf{A}\mathbf{x}) = 0$.

  If $\mathbf{x}$ is feasible in the primal, $\mathbf{y}$ is feasible in the dual, and $\mathbf{x}^{\mathrm{T}}(\mathbf{c} - \mathbf{A}^{\mathrm{T}}\mathbf{y}) = \mathbf{y}^{\mathrm{T}}(\mathbf{b} - \mathbf{A}\mathbf{x}) = 0$, then $\mathbf{x}$ and $\mathbf{y}$ are optimal for their respective problems.

Relations between primal and dual optimal solutions

primal (dual) problem	$\Longleftrightarrow$	dual (primal) problem
unique and non-degenerate solution	$\Longleftrightarrow$	unique and non-degenerate solution
unbounded solution	$\Longrightarrow$	no feasible solutions
no feasible solutions	$\Longrightarrow$	unbounded solution **or** no feasible solutions
degenerate solution	$\Longleftrightarrow$	alternative solutions

Exercises on duality

HOMEWORK!

- Formulate and solve graphically the dual of:

$$
\begin{array}{rrrrrl}
\text{minimize} & z = & 6x_1 & +3x_2 & +x_3 & \\
\text{subject to} & & 6x_1 & -3x_2 & +x_3 & \geq 2 \\
 & & 3x_1 & +4x_2 & +x_3 & \geq 5 \\
 & & & & x_1, x_2, x_3 & \geq 0
\end{array}
$$

- Then find the optimal primal solution

- Verify that the dual of the dual equals the primal