MVE165/MMG630, Applied Optimization
Lecture 6
Integer linear programming theory and algorithms

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Methods for ILP: Overview (Ch. 14.1)

- Enumeration
  - Implicit enumeration: Branch–and–bound
- Relaxations
  - Decomposition methods: Solve simpler problems repeatedly
  - Add valid inequalities to an LP – "cutting plane methods"
  - Lagrangian relaxation
- Heuristic algorithms – optimum *not* guaranteed
  - "Simple" rules $\Rightarrow$ feasible solutions
  - Construction heuristics
  - Local search heuristics

Relaxations and feasible solutions (Ch. 14.2)

- Consider a minimization integer linear program (ILP):

$$[\text{ILP}] \qquad z^* = \begin{array}{rrcll} \min & \mathbf{c}^{\mathrm{T}}\mathbf{x} & & & \\ \text{subject to} & \mathbf{A}\mathbf{x} & \leq & \mathbf{b} & \\ & \mathbf{x} & \geq & \mathbf{0} & \text{and integer} \end{array}$$

- The feasible set $X = \{\mathbf{x} \in Z^n_+ \,|\, \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$ is *non*-convex
- How prove that a solution $\mathbf{x}^* \in X$ is optimal?
- We cannot use strong duality/complementarity as for linear optimization (where X is polyhedral $\Rightarrow$ convexity)
- Bounds on the optimal value
  - Optimistic estimate $\underline{z} \leq z^*$ from a *relaxation* of ILP
  - Pessimistic estimate $\bar{z} \geq z^*$ from a *feasible solution* to ILP
- Goal: Find "good" feasible solution and tight bounds for z^*:

$$\bar{z} - \underline{z} \leq \varepsilon \text{ and } \varepsilon > 0 \text{ "small"}$$

Optimistic estimates of z^* from relaxations

- Either: Enlarge the set X by removing constraints
- Or: Replace $\mathbf{c}^{\mathrm{T}}\mathbf{x}$ by an underestimating function f, i.e., such that $f(\mathbf{x}) \leq \mathbf{c}^{\mathrm{T}}\mathbf{x}$ for all $\mathbf{x} \in X$
- Or: Do both

$\Rightarrow$ solve a *relaxation* of (ILP)

- Example (enlarge X):

  $X = \{\mathbf{x} \geq \mathbf{0} \mid \mathbf{A}\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \text{ integer } \}$ and
  $X^{\mathrm{LP}} = \{\mathbf{x} \geq \mathbf{0} \mid \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$

$$\Rightarrow \quad z^{\mathrm{LP}} = \min_{\mathbf{x} \in X^{\mathrm{LP}}} \mathbf{c}^{\mathrm{T}}\mathbf{x}$$

- It holds that $z^{\mathrm{LP}} \leq z^*$ since $X \subseteq X^{\mathrm{LP}}$

Relaxation principles that yield more tractable problems

- *Linear programming relaxation*
  Remove integrality requirements (enlarge X)
- *Combinatorial relaxation*
  E.g. remove subcycle constraints from asymmetric TSP $\Rightarrow$ min-cost assignment (enlarge X)
- *Lagrangean relaxation*
  Move "complicating" constraints to the objective function, with penalties for infeasible solutions; then find "optimal" penalties (enlarge X and find $f(\mathbf{x}) \leq \mathbf{c}^{\mathrm{T}}\mathbf{x}$)

Tight bounds

- Suppose that $\bar{\mathbf{x}} \in X$ is a feasible solution to ILP (min-problem) and that $\underline{\mathbf{x}}$ solves a relaxation of ILP
- Then
$$\underline{z} := \mathbf{c}^{\mathrm{T}}\underline{\mathbf{x}} \leq z^* \leq \mathbf{c}^{\mathrm{T}}\bar{\mathbf{x}} =: \bar{z}$$
- $\underline{z}$ is an *optimistic* estimate of z^*
- $\bar{z}$ is a *pessimistic* estimate of z^*
- If $\bar{z} - \underline{z} \leq \varepsilon$ then the value of the solution candidate $\bar{\mathbf{x}}$ is at most ε from the optimal value z^*
- Efficient solution methods for ILP combine relaxation and heuristic methods to find tight bounds (small $\varepsilon \geq 0$)

Branch–&–Bound algorithms (B&B) (Ch. 15)

$$\text{[ILP]} \qquad z^* = \min_{\mathbf{x} \in X} \mathbf{c}^{\mathrm{T}}\mathbf{x}, \qquad X \subset Z^n$$

- Divide–and–conquer: a general principle to partition and search the feasible space
- Branch–&–Bound: Divide–and–conquer for finding *optimal* solutions to optimization problems with integrality requirements
- Can be adapted to different types of models
- Can be combined with other (e.g. heuristic) algorithms
- Also called implicit enumeration and tree search
- *Idea:* Enumerate all feasible solutions by a successive partitioning of X into a family of subsets
- Enumeration organized in a tree using graph search; it is made implicit by utilizing approximations of z^* from relaxations of [ILP] for cutting off branches of the tree

Branch–&–bound for ILP: Main concepts

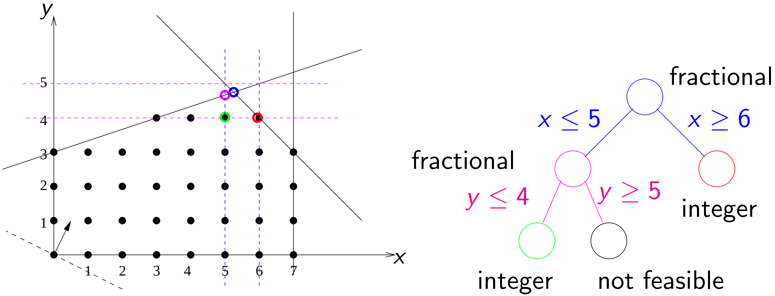
- Relaxation: a simplification of [ILP] in which some constraints are removed
  - Purpose: to get simple (polynomially solvable) (node) subproblems, and optimistic approximations of z^*.
  - Examples: remove integrality requirements, remove or Lagrangean relax complicating (linear) constraints (e.g. sub-tour constraints)
- Branching strategy: rules for partitioning a subset of X
  - Purpose: exclude the solution to a relaxation if it is not feasible in [ILP]; corresponds to a *partitioning* of the feasible set
  - Examples: Branch on fractional values, subtours, etc.

B&B: Main concepts (continued)

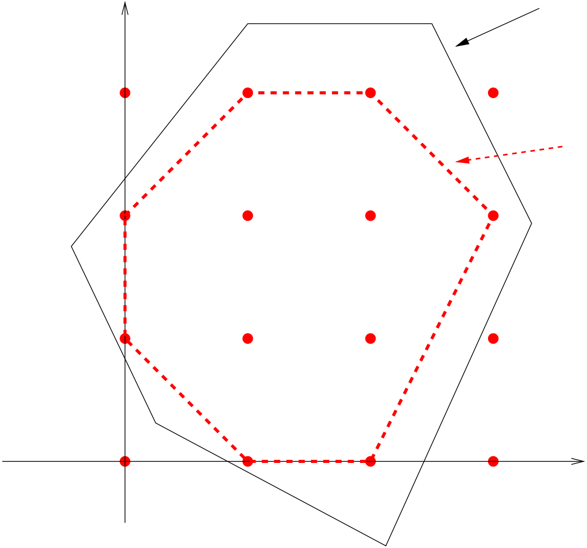
- Tree search strategy: defines the order in which the nodes in the B&B tree are created and searched
  - Purpose: quickly find good feasible solutions; limit the size of the tree
  - Examples: depth-, breadth-, best-first.
- Node cutting criteria: rules for deciding when a subset should not be further partitioned
  - Purpose: avoid searching parts of the tree that cannot contain an optimal solution
  - Cut off a node if the corresponding node subproblem has
    - no feasible solution, or
    - an optimal solution that is feasible in [ILP], or
    - an optimal objective value that is worse (higher) than that of any known feasible solution

ILP: Solution by the branch–and–bound algorithm

- Relax integrality requirements ⇒ linear (continuous) problem
- B&B tree: branch over fractional variable values



Good and ideal formulations (Ch. 14.3)



$\mathbf{Ax} \leq \mathbf{b}$

Ideal since all extreme points are integral

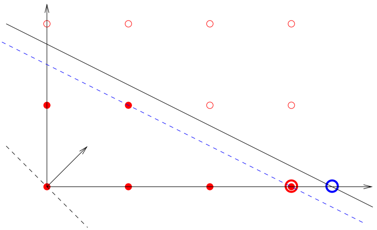
Linear program has integer extreme points

Cutting planes: A very small example

- Consider the following ILP:

$$\min\{-x_1 - x_2 : 2x_1 + 4x_2 \leq 7, x_1, x_2 \geq 0 \text{ and integer}\}$$

- ILP optimal solution: $z = -3$, $\mathbf{x} = (3, 0)$
- LP (continuous relaxation) optimum: $z = -3.5$, $\mathbf{x} = (3.5, 0)$
- Generate a simple cut:
  "Divide the constraint" by 2
  and round the RHS down
  $x_1 + 2x_2 \leq 3.5 \Rightarrow x_1 + 2x_2 \leq 3$
- Adding this cut to the continuous relaxation yields the optimal ILP solution

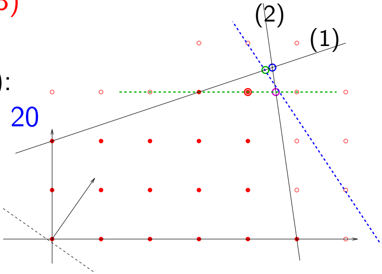


Cutting planes: valid inequalities (Ch. 14.4)

- Consider the ILP

$$\begin{array}{llcll} \max & 7x_1 + 10x_2 & & & \\ \text{subject to} & -x_1 + 3x_2 & \leq & 6 & (1) \\ & 7x_1 + x_2 & \leq & 35 & (2) \\ & x_1, x_2 & \geq & 0, \text{ integer} & \end{array}$$

- LP optimum: $z = 66.5$, $\mathbf{x} = (4.5, 3.5)$
- ILP optimum: $z = 58$, $\mathbf{x} = (4, 3)$
- Generate a VI by "adding" the two constraints (1) and (2): $6x_1 + 4x_2 \leq 41 \Rightarrow 3x_1 + 2x_2 \leq 20$ $\Rightarrow \mathbf{x} = (4.36, 3.45)$
- Generate a VI by "7·(1)+(2)": $22x_2 \leq 77 \Rightarrow x_2 \leq 3$ $\Rightarrow \mathbf{x} = (4.57, 3)$



Cutting plane algorithms (iterativley better approximations of the convex hull) (Ch. 14.5)

- Choose a suitable mathematical formulation of the problem

1. Solve the linear programming (LP) relaxation
2. If the solution is integer, Stop. An optimal solution is found
3. Add one or several *valid inequalities* that cut off the fractional solution *but none of the integer solutions*
4. Resolve the new problem and go to step 2.

- *Remark:* An inequality in higher dimensions defines a *hyper-plane*; therefore the name cutting *plane*

About cutting plane algorithms

- Problem: It may be necessary to generate VERY MANY cuts
- Each cut should also pass through at least one integer point $\Rightarrow$ faster convergence
- Methods for generating valid inequalities
  - Chvatal-Gomory cuts (combine constraints, make beneficial roundings of LHS and RHS)
  - Gomory's method: generate cuts from an optimal simplex basis (Ch. 14.5.1)
- Pure cutting plane algorithms are usually less efficient than branch–&–bound
- In commercial solvers (e.g. CPLEX), cuts are used to help (presolve) the branch–&–bound algorithm
- For problems with specific structures (e.g. TSP and set covering) problem specific classes of cuts are used

Lagrangian relaxation ($\Rightarrow$ optimistic estimates of z^*) (Ch. 17.1–17.2)

- Consider a minimization integer linear program (ILP):

$$
\begin{array}{llrlll}
\text{[ILP]} & z^* = & \min & \mathbf{c}^{\mathrm{T}}\mathbf{x} & & \\
& & \text{subject to} & \mathbf{A}\mathbf{x} & \leq \mathbf{b} & (1) \\
& & & \mathbf{D}\mathbf{x} & \leq \mathbf{d} & (2) \\
& & & \mathbf{x} & \geq \mathbf{0} \text{ and integer} &
\end{array}
$$

- Assume that the constraints (1) are complicating (subtour eliminating constraints for TSP, e.g.)
- Define the set $X = \{\mathbf{x} \in Z_+^n \,|\, \mathbf{D}\mathbf{x} \leq \mathbf{d}\}$
- Remove the constraints (1) and add them—with penalty parameters $\mathbf{v}$—to the objective function

$$h(\mathbf{v}) = \min_{\mathbf{x} \in X} \left\{ \mathbf{c}^{\mathrm{T}}\mathbf{x} + \mathbf{v}^{\mathrm{T}}(\mathbf{A}\mathbf{x} - \mathbf{b}) \right\} \qquad (3)$$

Weak duality of Lagrangian relaxations

Theorem: For any $\mathbf{v} \geq \mathbf{0}$ it holds that $h(\mathbf{v}) \leq z^*$.

Proof: Let $\bar{\mathbf{x}}$ be feasible in [ILP] $\Rightarrow \bar{\mathbf{x}} \in X$ and $\mathbf{A}\bar{\mathbf{x}} \leq \mathbf{b}$. It then holds that

$$h(\mathbf{v}) = \min_{\mathbf{x} \in X} \left\{ \mathbf{c}^{\mathrm{T}}\mathbf{x} + \mathbf{v}^{\mathrm{T}}(\mathbf{A}\mathbf{x} - \mathbf{b}) \right\} \leq \mathbf{c}^{\mathrm{T}}\bar{\mathbf{x}} + \mathbf{v}^{\mathrm{T}}(\mathbf{A}\bar{\mathbf{x}} - \mathbf{b}) \leq \mathbf{c}^{\mathrm{T}}\bar{\mathbf{x}}.$$

Since an optimal solution $\mathbf{x}^*$ to [ILP] is also feasible, it holds that

$$h(\mathbf{v}) \leq \mathbf{c}^{\mathrm{T}}\mathbf{x}^* = z^*.$$

□

$\Rightarrow$ $h(\mathbf{v})$ is a *lower bound* on the optimal value z^* for any $\mathbf{v} \geq \mathbf{0}$

- The best lower bound is given by

$$h^* = \max_{\mathbf{v} \geq \mathbf{0}} h(\mathbf{v}) = \max_{\mathbf{v} \geq \mathbf{0}} \left\{ \min_{\mathbf{x} \in X} \left\{ \mathbf{c}^{\mathrm{T}}\mathbf{x} + \mathbf{v}^{\mathrm{T}}(\mathbf{A}\mathbf{x} - \mathbf{b}) \right\} \right\}$$

Tractable Lagrangian relaxations

- Special algorithms for minimizing the Lagrangian dual function h exist (e.g., subgradient optimization, Ch. 17.3)
- h is always concave but typically nondifferentiable
- For each value of $\mathbf{v}$ chosen, a *subproblem* (3) must be solved
- For general ILP's: typically a non-zero duality gap $h^* < z^*$
- The Lagrangian relaxation bound is never worse that the linear programming relaxation bound, i.e. $z^{\mathrm{LP}} \leq h^* \leq z^*$
- If the set X has the integrality property (i.e., X^{LP} has integral extreme points) then $h^* = z^{\mathrm{LP}}$
- Choose the constraints ($\mathbf{Ax} \leq \mathbf{b}$) to dualize such that the relaxed problem (3) is computationally tractable but still does not possess the integrality property

An ILP Example

[HOMEWORK]

Find optimistic and pessimistic bounds for the following ILP example using the branch–&–bound algorithm, a cutting plane algorithm, and Lagrangean relaxation.

$$\begin{array}{lrcl} \max & 5x_1 + 4x_2 & & \\ \text{s.t.} & x_1 + x_2 & \leq & 5 \\ & 10x_1 + 6x_2 & \leq & 45 \\ & x_1, x_2 & \geq & 0 \text{ and integer} \end{array}$$

The linear programming optimal solution is given by $z = 23.75$, $x_1 = 3.75$ and $x_2 = 1.25$