MVE165/MMG631
Linear and Integer Optimization with Applications
Lecture 1
Introduction; course map; operations research; modelling; graphic solution

Ann-Brith Strömberg

2014–03–18

Staff

- **Examiner and main lecturer**
  - Ann-Brith Strömberg (anstr@chalmers.se, room L2087)
- **Course assistant**
  - Zuzana Šabartová (zuzana@chalmers.se, room L2099)
- **Problem solving sessions**
  - Zuzana Šabartová (zuzana@chalmers.se, room L2099)
  - Kin Cheong Sou (kincheong@chalmers.se, room L2100)
- **Guest lecturers**
  - Emil Gustavsson (Mathematical Sciences)
  - Ola Carlson (Energy and Environment)

Course homepage and information

- **Course homepage**
  - `www.math.chalmers.se/Math/Grundutb/CTH/mve165/1314`
  - Details, information on assignmnents and computer exercises, deadlines, lecture notes, exercises etc
  - Will be updated with new information every week

- **PingPong**
  - `https://pingpong.gate.chalmers.se`
  - Software download (AMPL & CPLEX)
  - Hand-in of assignments

Organization

- **Lectures** – mathematical optimization theory
- **Computer exercise** – learn how to use software solvers
- **Guest lectures** – applications of optimization $\Rightarrow$ assignments
- **Assignments** – modelling, use solvers, analyze solutions, write reports, opposition & oral presentation
- *Assignment work should be done in groups of $\leq$ two persons*
- Define your project groups on the PingPong page of MVE165/MMG631
- The name of the project group must be: *"FirstName1 Surname1 - FirstName2 Surname2"*
- GU and PhD students not having PingPong-entries: *sign a list*
- Computers are reserved in F-T7203 on Tuesdays and Wednesdays, 13.15–15.00. (Week 12: Tuesday 13.15-15.00 (F-T7203) and Friday 13.15-17.00 (MVF24-25)). Teachers will be present only when indicated in the course plan on the home page.

Software

- **A computer exercise on linear optimization and software** is found on the homepage. You are highly recommended to perform it to prepare for the assignments.
- **AMPL-packages** to install on your own computer (linux, mac, windows) is available via Ping-pong. Read the agreement text! Also presented on Lecture 2.
- **Matlab**
- **A java-applet for learning the branch-and-bound algorithm**

Course evaluation process

- Questionnaires will be sent out to all students registered for the course after the exam week
- The examiner calls the course representatives to the introductory, (recommended) middle, and final meetings
- Course representatives
  - ??
  - ??

Literature

- **Main course book**:
  - English version: Optimization (2010)
  - Swedish version: Optimeringslära (2008)

  by J. Lundgren, M. Rönnqvist, and P. Värbrand. Studentlitteratur.

- **Exercise book**:
  - English version: Optimization Exercises (2010)
  - Swedish version: Optimeringslära Övningsbok (2008)

  by M. Henningsson, J. Lundgren, M. Rönnqvist, and P. Värbrand. Studentlitteratur.

- Cremona/Studentlitteratur/Adlibris/...

- Also some **hand-outs** (denoted in the lecture notes)

Examination requirements

- Perform **three project assignments** in groups of two students
  - For Assignment 3 there will be two alternatives
- **Written reports** of three assignments
- **A written opposition** to another group's report of Assignment 2
- **An oral presentation** of Assignment 3
- **Presence** at one full oral presentation session
- *To be able to receive a grade higher than 3 or G, the written reports and opposition as well as the oral presentation must be of high quality. Students aiming at grade 4, 5, or VG must also pass an oral exam*

Overview of the lectures and course contents

Mathematical subjects

- Linear optimization models, modelling, theory, solution methods, and sensitivity analysis
- Discrete optimization models, properties, and solution methods
- Optimization problems that can be modelled as flows in networks, theory, and solution methods
- Multi-objective optimization
- Overview of non-linear optimization models, properties, and solution methods
- Mixes of the above

Activities

- Applications of optimization
- Mathematical modelling
- Theory – mathematical properties of models
- Solution techniques – algorithms
- Software solvers

Optimization: "Do something as good as possible"

- **Something:** Which are the decision alternatives?
  - $x_{ij} = \begin{cases} 1 & \text{if customer } j \text{ is visited } \textit{directly} \text{ after customer } i \\ 0 & \text{else} \end{cases}$
  - $y_{ik} = \begin{cases} 1 & \text{if customer } i \text{ is visited by vehicle } k \\ 0 & \text{else} \end{cases}$
- **Possible:** What restrictions are there?
  - Each customer should be visited exactly once
  - Time windows, transport needs and capacity, pick up at one customer – deliver at another, different types of vehicles, ...
- **Good:** What is a relevant optimization criterion?
  - Minimize the total distance / travel time / emissions / waiting time / ...

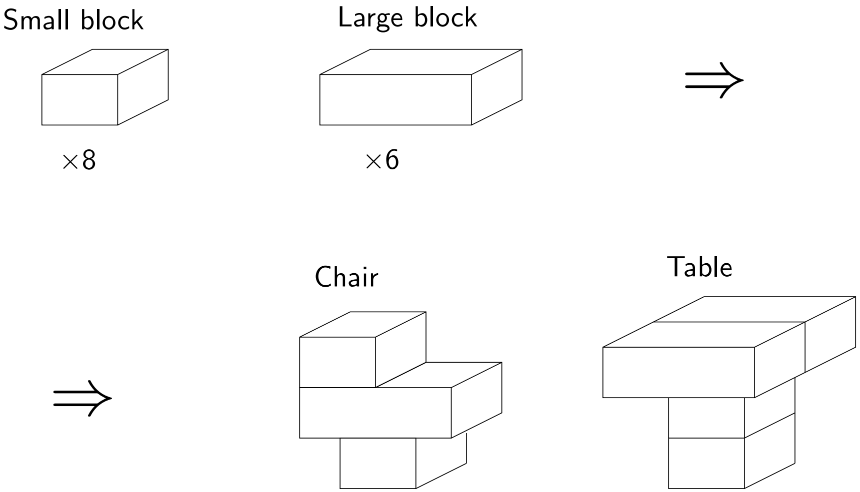
The classical travelling salesperson problem

- Variants of routing problems: e.g., refrigerated goods, transportation service for disabled persons, school buses, ...

Examples of application areas

- **Logistics: production and transport**
  - Optimize routes for transports, snow removal, school buses, ...
  - Location of stores
  - Planning of wood cut and transports
  - Packing of containers
  - Production planning and scheduling
- **Energy**
  - Energy production planning
  - Investment in energy production technology
  - Location of power plants and infrastructure
- **Finance**
  - Financial risk management
  - Portfolio optimization
  - Investment planning
- **Medicine**
  - Compute radiation directions/intensities for cancer treatment
  - Reconstruct images from x-ray measurements

A manufacturing example: Produce tables and chairs from two types of blocks



A manufacturing example, continued

- A chair is assembled from one large and two small blocks
- A table is assembled from two blocks of each size
- Only 6 large and 8 small blocks are avaliable
- A table is sold at a revenue of 1600:-
- A chair is sold at a revenue of 1000:-
- Assume that all items produced can be sold and determine an optimal production plan

A mathematical optimization model

- **Something** – What decision alternatives? $\Rightarrow$ Variables

$x_1 =$ number of tables produced and sold
$x_2 =$ number of chairs produced and sold

- **Possible** – What restrictions? $\Rightarrow$ Constraints
  - Maximum supply of large blocks: 6

$$\boxed{2x_1 + x_2 \leq 6}$$

  - Maximum supply of small blocks: 8

$$\boxed{2x_1 + 2x_2 \leq 8}$$

  - Physical restrictions (also: x_1, x_2 integral)

$$\boxed{x_1, x_2 \geq 0}$$

- **Good** – Relevant optimization criterion? $\Rightarrow$ Objective function
  - Maximize the total revenue

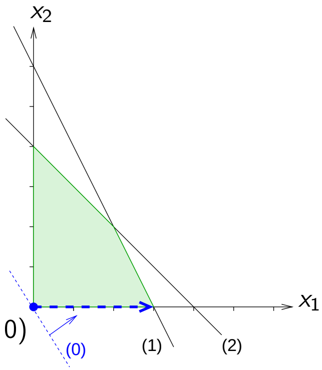
$$\boxed{1600x_1 + 1000x_2 \rightarrow \max}$$

Solve the model using LEGO and marginal values

- Start at no production:
  $x_1 = x_2 = 0$
  Use the "best marginal profit" to choose the item to produce
  - x_1 has the highest marginal profit (1600:-/table)
    $\Rightarrow$ produce as many tables as possible
  - At $x_1 = 3$: no more large blocks left

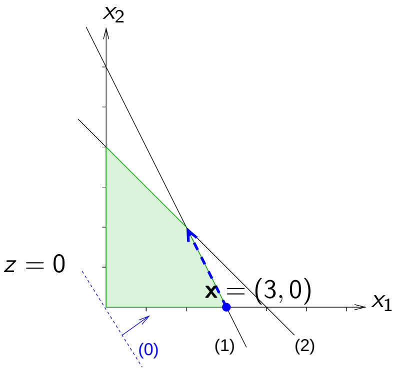
$z = 0$

$\mathbf{x} = (0, 0)$



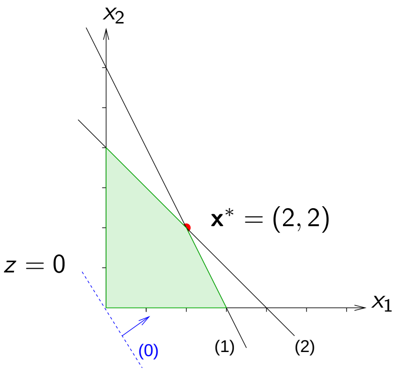
Solve the model using LEGO and marginal values

- The marginal value of x_2 is now 200:- since taking apart one table (-1600:-) yields two chairs ($+2000$:-) $\Rightarrow$ 400:-/2 chairs
  - Increase x_2 maximally $\Rightarrow$ decrease x_1
  - At $x_1 = x_2 = 2$: no more small blocks



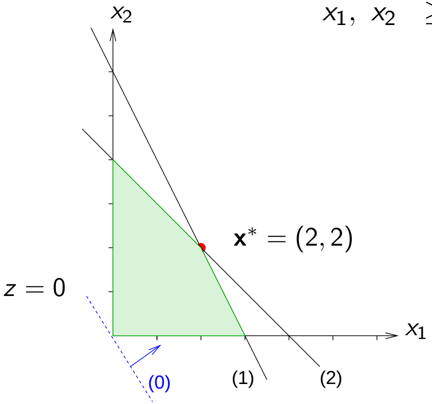
Solve the model using LEGO and marginal values

- The marginal value of x_1 is negative (to build one more table one has to take apart two chairs $\Rightarrow -400$:-)
  The marginal value of x_2 is -600:- (to build one more chair one table must be taken apart)
  $\Longrightarrow$ Optimal solution:
  $x_1 = x_2 = 2$



Geometric solution of the model

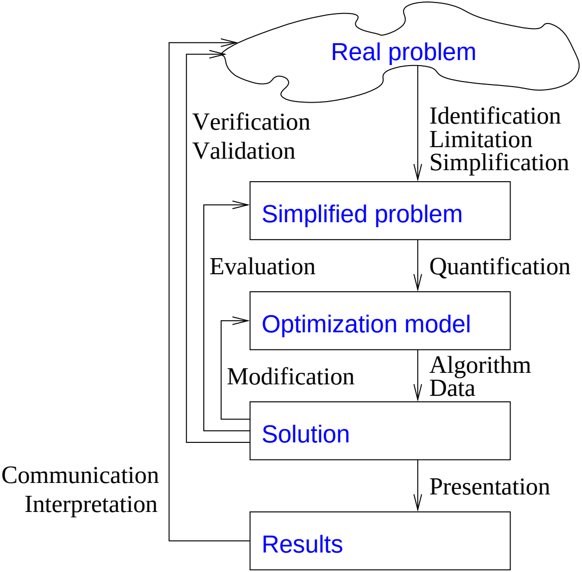
$$
\begin{array}{lrcrcrcl}
\text{maximize} & z & = & 1600x_1 & + & 1000x_2 & & & (0)\\
\text{subject to} & & & 2x_1 & + & x_2 & \leq & 6 & (1)\\
 & & & 2x_1 & + & 2x_2 & \leq & 8 & (2)\\
 & & & & & x_1,\ x_2 & \geq & 0 &
\end{array}
$$



Operations Research (OR) (Swedish: Operationsanalys)

- Scientific view on problem solving regarding complex systems
- *"OR means to prepare a foundation for rational decisions by utilizing systematic scientific methods and — when possible and meaningful — by utilizing quantitative models"*
- The problem is considered as a system of components which cooperate and influence each other
- The activities studied are described by models, used to
  - better understand the depicted system,
  - understand the consequences of different decisions, and
  - choose the "best" alternative due to some criterion.

The process of optimization

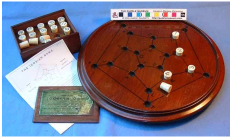


History of Operations Research

- During world war II decision problems became systematically treated: Operations Research
- After the war: use of operations research for civil operations
- The ideas spread to many countries
- Early operations research include inventory planning

A few moments in optimization history

- Euler (1735): Seven bridges of Köningsberg (graph theory)
- Gauss (1777–1855): 1st optimization technique – steepest descent
- W.R. Hamilton (1857): "icosian game" $\Rightarrow$ the travelling salesperson problem (Hamilton cycle)



- L.V. Kantorovich (1939): A linear model for optimization of plywood manufactoring and an algorithm for its solution
- George B. Dantzig (1947): Linear programming – the simplex algorithm (exponential time)
  - Program $\Leftrightarrow$ military training and logistics schedules
- Kantorovich and Koopmans (1975): Nobel Memorial Prize in Economic Sciences
- N. Karmarkar (1984) algorithm for linear programming (polynomial time)

Optimization modelling: Electric power capacity expansion

- An electric utility will install two generators ($j = 1, 2$) with different fixed and operating costs, to meet the demand within its service region.
- Each day is divided into three *parts* ($i = 1, 2, 3$) of equal duration, during which the demand, d_i, takes a *base*, *medium*, or *peak* value, respectively.
- The *fixed cost* per unit capacity of generator j is amortized over its lifetime and amounts to c_j per day.
- The *operating cost* of generator j during the ith part of the day is f_{ij}.
- The *availability* of generator j is $a_j \in [0, 1]$, $j = 1, 2$.
- If the demand during the ith part of the day cannot be served due to lack of capacity, *additional capacity* must be purchased at a cost of g_i per unit.
- The *capacity* of each generator j is required to be at least b_j.

Define the decision variables

- Let the variables x_j, $j = 1, 2$, represent the installed capacity of generator j.
- Let the variables y_{ij} denote the operating level of generator j during the ith part of the day.
- Let the variable w_i denote the capacity that needs to be purchased, in order to satisfy unmet demand during the ith part of the day.
- We interpret availability to mean that the operating level of generator j, at any given time is *at most* $a_j x_j$.

General mathematical optimization models

- $$\left[\begin{array}{lll} \text{minimize or maximize} & f(x_1,\ldots,x_n) & \\ \text{subject to} & g_i(x_1,\ldots,x_n) \left\{\begin{array}{c} \leq \\ = \\ \geq \end{array}\right\} b_i, & i=1,\ldots,m \end{array}\right]$$

- $x_1,\ldots,x_n$ are the decision variables
- f and $g_1,\ldots,g_m$ are given functions of the decision variables
- $b_1,\ldots,b_m$ are specified constant parameters
- The functions can be nonlinear, e.g. quadratic, exponential, logarithmic, non-analytic, ...
- In general, linear forms are more tractable than non-linear

Linear optimization models (programs)

- The capacity expansion model is a linear program (LP), i.e., all relations are described by linear forms
- A general linear program:

$$\left[\begin{array}{llccl} \text{min or max} & c_1x_1 + \ldots + c_nx_n & & & \\ \text{subject to} & a_{i1}x_1 + \ldots + a_{in}x_n & \left\{\begin{array}{c} \leq \\ = \\ \geq \end{array}\right\} & b_i, & i = 1, \ldots, m \\ & x_j & \geq & 0, & j = 1, \ldots, n \end{array}\right]$$

- The non-negativity constraints on x_j, $j = 1, \ldots, n$ are not necessary, but usually assumed (reformulation always possible)

Discrete/integer/binary modelling

- A variable is called *discrete* if it can take only a countable set of values, e.g.,
  - Continuous variable: $x \in [0, 8] \iff 0 \leq x \leq 8$
  - Discrete variable: $x \in \{0, 4.4, 5.2, 8.0\}$
  - *Integer* variable: $x \in \{0, 1, 4, 5, 8\}$
- A *binary* variable can only take the values 0 or 1, i.e., all or nothing

  E.g., a wind-mill can produce electricity only if it is built
  - Let $y = 1$ if the mill is built, otherwise $y = 0$
  - Capacity of a mill: C
  - Production $x \leq Cy$ (also limited by wind force etc.)
- In general, models with only continuous variables are more tractable than models with integrality/discrete requirements on the variables, but exceptions exist! More about this later.