MVE165/MMG631
Linear and integer optimization with applications
Lecture 12
Maximum flows and minimum cost flows—models and algorithms

Ann-Brith Strömberg

2014–05–09

Maximum flow models (Ch. 8.6)

- Consider a district heating network with pipelines that transports energy (in the form of hot water) from a number of sources to a number of destinations
- The network has several branches and junctions
- Pipe segment (i, j) has a maximum capacity of K_{ij} units of flow per time unit
- A pipe can be one- or bidirectional
- What is the maximum total amount of flow per time unit through this network?
- Another application of the maximum flow model: evacuation of buildings (also time dynamics)

LP model for maximum flow problems

- Let x_{ij} denote the amount of flow through pipe segment (i, j) (flow direction $i \to j$)
- Let v denote the *total flow* from the source to the destination
- *Graph*: $G = (V, A, K)$ (nodes, directed arcs, arc capacities) (an undirected edge is here represented by two directed arcs)

$$
\begin{array}{lrcll}
\max & v, & & & \\
\text{s.t.} & \displaystyle\sum_{j:(s,j)\in A} (-x_{sj}) + v & = & 0, & \\
 & \displaystyle\sum_{j:(j,t)\in A} x_{jt} - v & = & 0, & \\
 & \displaystyle\sum_{i:(i,k)\in A} x_{ik} + \sum_{j:(k,j)\in A} (-x_{kj}) & = & 0, & k \in V \setminus \{s, t\} \\
 & x_{ij} & \leq & K_{ij}, & (i, j) \in A \\
 & x_{ij} & \geq & 0, & (i, j) \in A
\end{array}
$$

Draw!!

A solution method for maximum flow problems (Edmonds & Karp, 1972)

1. Let $k := 0$, $v := 0$, $x_{ij}^0 := 0$, and $u_{ij}^0 := K_{ij}$, $(i,j) \in A$.
2. Find a *maximum capacity* path $P^k \subset A$ from s to t (modified shortest path algorithm). The capacity of P^k is
   $\hat{u}^k := \min \left\{ \min\{u_{ij}^k \mid (i,j) \in P^k\}; \min\{x_{ij}^k \mid (j,i) \in P^k\} \right\}$.
   If $\hat{u}^k = 0$, go to step 4.
3. Update the flows $x_{ij}^{k+1} := \begin{cases} x_{ij}^k + \hat{u}^k, & \text{if } (i,j) \in P^k, \\ x_{ij}^k - \hat{u}^k, & \text{if } (j,i) \in P^k, \\ x_{ij}^k, & \text{otherwise,} \end{cases}$
   the capacities $u_{ij}^{k+1} := \begin{cases} u_{ij}^k - \hat{u}^k, & \text{if } (i,j) \in P^k, \\ u_{ij}^k + \hat{u}^k, & \text{if } (j,i) \in P^k, \\ u_{ij}^k, & \text{otherwise,} \end{cases}$
   and the total flow $v^{k+1} := v^k + \hat{u}^k$. Let $k := k+1$, go to step 2.
4. The maximum total flow equals v^k.
   The flow solution is given by x_{ij}^k, $(i,j) \in A$.

LP dual of the maximum flow model

$$
\begin{array}{llrcll}
\text{[Primal]} & \max & v, & & & \\
& \text{s.t.} & \displaystyle\sum_{j:(s,j)\in A} (-x_{sj}) + v & = & 0, & \\
& & \displaystyle\sum_{j:(j,t)\in A} x_{jt} - v & = & 0, & \\
& & \displaystyle\sum_{i:(i,k)\in A} x_{ik} + \sum_{j:(k,j)\in A} (-x_{kj}) & = & 0, & k \in V \setminus \{s,t\} \\
& & 0 \leq x_{ij} & \leq & K_{ij}, & (i,j) \in A
\end{array}
$$

$$
\begin{array}{llrcll}
\text{[Dual]} & \min & \displaystyle\sum_{(i,j)\in A} K_{ij}\gamma_{ij}, & & & \\
& \text{s.t.} & -\pi_i + \pi_j + \gamma_{ij} & \geq & 0, & (i,j) \in A \\
& & \pi_s - \pi_t & = & 1, & \\
& & \pi_k & & \text{free}, & k \in V, \\
& & \gamma_{ij} & \geq & 0, & (i,j) \in A
\end{array}
$$

Draw!!

Maximum flow – Minimum cut theorem

- An (s,t)-*cut* is a set of arcs which, when deleted, interrupt all flow in the network between the source s and the sink t
- The *cut capacity* equals the sum of capacities on all the arcs through the (s,t)-cut
- Finding the minimum (s,t)-cut is equivalent to solving the dual of the maximum flow problem
- *Weak duality theorem: Each feasible flow x_{ij}, $(i,j) \in A$, yields a lower bound on v^*. The capacity of each (s,t)-cut yields an upper bound on v^*.*
- *Strong duality theorem: value of maximum flow = capacity of minimum cut*

Optimal dual solution – minimum cut

- Optimal values of the dual variables:

$$\gamma_{ij} = \begin{cases} 1, & \text{if arc } (i,j) \text{ passes through the minimum cut,} \\ 0, & \text{otherwise.} \end{cases}$$

$$\pi_k = \begin{cases} 1, & \text{if node } k \text{ can be reached from } s, \\ 0, & \text{otherwise.} \end{cases}$$

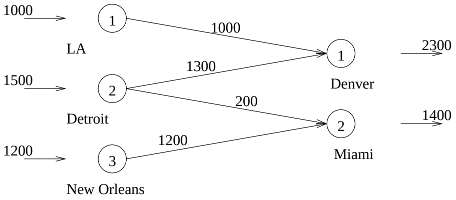
- How is the minimum cut found using the Edmonds & Karp algorithm?

Transportation models: An example

- MG Auto has three plants, LA, Detroit, New Orleans, and two distribution centers, Denver and Miami
- Capacities of the plants: 1000, 1500, and 1200 cars
- Demands at distributions centers: 2300 and 1400 cars
- Transportation cost per car between plants and centers:

	Denver	Miami
LA	\$80	\$215
Detroit	\$100	\$108
New Orleans	\$102	\$68

- Find the cheapest shipping schedule to satisfy the demand



Linear programming formulation of MG Auto

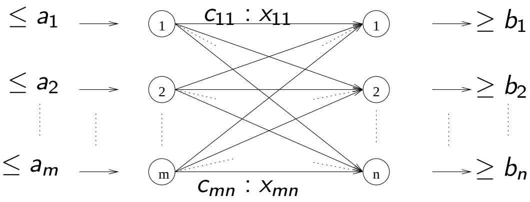
- Variables: x_{ij} = number of cars sent from plant i to distribution center j

$$\min z = 80x_{11} + 215x_{12} + 100x_{21} + 108x_{22} + 102x_{31} + 68x_{32}$$

$$\begin{array}{llllllllll}
\text{s.t.} & x_{11} & +x_{12} & & & & & \leq & 1000 & \text{(LA)} \\
& & & x_{21} & +x_{22} & & & \leq & 1500 & \text{(Detr)} \\
& & & & & x_{31} & +x_{32} & \leq & 1200 & \text{(NO)} \\
& & & & & & & & & \\
& x_{11} & & +x_{21} & & +x_{31} & & \geq & 2300 & \text{(Den)} \\
& & x_{12} & & +x_{22} & & +x_{32} & \geq & 1400 & \text{(Mi)} \\
& & & & & & & & & \\
& x_{11}, & x_{12}, & x_{21}, & x_{22}, & x_{31}, & x_{32} & \geq & 0 &
\end{array}$$

Definition of the transportation model

- m sources and n destinations $\Leftrightarrow$ **nodes**
- a_i = amount of supply at source (node) i, $i = 1, \ldots, m$
- b_j = amount of demand at destination (node) j, $j = 1, \ldots, n$
- *Arc* (i,j) $\Leftrightarrow$ connection from source i to destination j
- c_{ij} = cost per unit of flow on arc (i,j)
- *Variables:* x_{ij} = amount of goods shipped on arc (i,j)
- *Objective:* find $x_{ij} \geq 0$ such that the total cost is minimized while satisfying all supply and demand restrictions



Linear programming transportation model

$$
\begin{aligned}
\min z := \quad & \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \\
\text{s.t.} \quad & \sum_{j=1}^{n} x_{ij} \leq a_i, \quad i = 1, \ldots, m \qquad \text{(supply)} \\
& \sum_{i=1}^{m} x_{ij} \geq b_j, \quad j = 1, \ldots, n \qquad \text{(demand)} \\
& x_{ij} \geq 0, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n
\end{aligned}
$$

- Feasible solutions exist *if and only if* $\boxed{\sum_i a_i \geq \sum_j b_j}$
- The constraint matrix has special properties (totally unimodular) $\Rightarrow$ extreme points of the feasible polyhedron are integer (Chapter 8.6.3)

A balanced transportation model

- What if total amount of demand $\neq$ total amount of supply? ($\sum_i a_i > \sum_j b_j$ (feasible) or $\sum_i a_i < \sum_j b_j$ (infeasible))

$$\boxed{\begin{array}{lrcll} \min z := & \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} & & & \\ \text{s.t.} & \sum_{j=1}^{n} x_{ij} & \leq & a_i, & i = 1, \dots, m \\ & \sum_{i=1}^{m} x_{ij} & \geq & b_j, & j = 1, \dots, n \\ & x_{ij} & \geq & 0, & i = 1, \dots, m, j = 1, \dots, n \end{array}}$$

$\Rightarrow$ Balance the model by dummy source $m+1$ or destination $n+1$

- Suppose $\sum_i a_i > \sum_j b_j \Rightarrow$ Let $b_{n+1} := \sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j$

$\Rightarrow$ Balanced transportation model: equality constraints

$$\boxed{\begin{array}{lrcll} \min z := & \sum_{i=1}^{m} \sum_{j=1}^{n+1} c_{ij} x_{ij} & & & \\ \text{s.t.} & \sum_{j=1}^{n+1} x_{ij} & = & a_i, & i = 1, \dots, m \\ & \sum_{i=1}^{m} x_{ij} & = & b_j, & j = 1, \dots, n+1 \\ & x_{ij} & \geq & 0, & i=1, \dots, m, j=1, \dots, n+1 \end{array}}$$

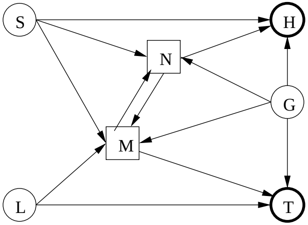
General minimum cost network flow problems

- A network consist of a set N of *nodes* linked by a set A of *arcs*
- A distance/cost c_{ij} is associated with each arc
- Each node i in the network has a net demand d_i
- Each arc carries an (unknown) amount of flow x_{ij} that is restricted by a maximum capacity $u_{ij} \in [0, \infty]$ and a minimum capacity $\ell_{ij} \in [0, u_{ij}]$
- The flow through each node must be *balanced*
- A network flow problem can be formulated as a linear program
- All extreme points of the feasible set are *integral* – due to the *unimodularity* property of the constraint matrix (see Ch. 8.6.3)

Minimum cost flow in a general network: Example

- Two paper mills: Holmsund and Tuna
- Three saw mills: Silje, Graninge and Lunden
- Two storage terminals: Norrstig and Mellansel

Facility	Supply (m^3)	Demand (m^3)
Silje	2400	
Graninge	1800	
Lunden	1400	
Holmsund		3500
Tuna		2100



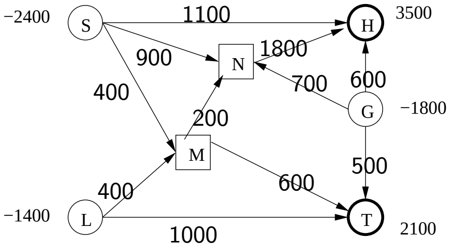
Minimum cost flow in a general network: Example

- Transportation opportunities:

From	To	Price/m^3	Capacity (m^3)
Silje	Norrstig	20	900
Silje	Mellansel	26	1000
Silje	Holmsund	45	1100
Graninge	Norrstig	8	700
Graninge	Mellansel	14	900
Graninge	Holmsund	37	600
Graninge	Tuna	22	600
Lunden	Mellansel	32	600
Lunden	Tuna	23	1000
Norrstig	Holmsund	11	1800
Norrstig	Mellansel	9	1800
Mellansel	Norrstig	9	1800
Mellansel	Tuna	9	1800

Minimum cost flow in a general network: Example

- Objective: Minimize transportation costs
- Satisfy demand
- Do not exceed the supply
- Do not exceed the transportation capacities
- An optimal solution



Minimum cost flow in a general network: Example

$$
\begin{array}{llcll}
\min z := & 20x_{SN} + 26x_{SM} + 45x_{SH} + 8x_{GN} + 14x_{GM} & & & \\
 & +37x_{GH} + 22x_{GT} + 32x_{LM} + 23x_{LT} + 11x_{NH} & & & \\
 & +9x_{NM} + 9x_{MN} + 9x_{MT} & & & \\
\text{subject to} & -x_{SN} - x_{SM} - x_{SH} & = & -2400 & \text{(Silje)} \\
 & -x_{GN} - x_{GM} - x_{GH} - x_{GT} & = & -1800 & \text{(Graninge)} \\
 & -x_{LM} - x_{LT} & = & -1400 & \text{(Lunden)} \\
 & x_{SN} + x_{GN} + x_{MN} - x_{NM} - x_{NH} & = & 0 & \text{(Norrstig)} \\
 & x_{SM} + x_{LM} + x_{GM} + x_{NM} - x_{MN} - x_{MT} & = & 0 & \text{(Mellansel)} \\
 & x_{SH} + x_{GH} + x_{NH} & = & 3500 & \text{(Holmsund)} \\
 & x_{GT} + x_{LT} + x_{MT} & = & 2100 & \text{(Tuna)} \\
 & 0 \leq x_{SN} & \leq & 900 & \\
 & 0 \leq x_{SM} & \leq & 1000 & \\
 & 0 \leq x_{SH} & \leq & 1100 & \\
 & 0 \leq x_{GN} & \leq & 700 & \\
 & 0 \leq x_{GM} & \leq & 900 & \\
 & 0 \leq x_{GH} & \leq & 600 & \\
 & 0 \leq x_{GT} & \leq & 600 & \\
 & 0 \leq x_{LM} & \leq & 600 & \\
 & 0 \leq x_{LT} & \leq & 1000 & \\
 & 0 \leq x_{NH} & \leq & 1800 & \\
 & 0 \leq x_{NM} & \leq & 1800 & \\
 & 0 \leq x_{MN} & \leq & 1800 & \\
 & 0 \leq x_{MT} & \leq & 1800 &
\end{array}
$$

- The columns $\mathbf{A}_j$ of the equality constraint matrix ($\mathbf{Ax} = \mathbf{b}$) have one 1-element, one -1-element; the remaining elements are 0 $\Rightarrow$ the matrix $\mathbf{A}$ is totally unimodular

Minimum cost flows in general networks: LP model

- $G = (N, A)$ is a network with nodes N and arcs A, $|N| = n$
- x_{ij} is the amount of flow on the arc from node i to node j,
- ℓ_{ij} and u_{ij} are lower and upper limits for the flow on arc (i, j),
- c_{ij} is the cost per unit of flow on arc (i, j), and
- d_i is the demand in node i

$$
\begin{array}{lrcll}
\min & \displaystyle\sum_{(i,j)\in A} c_{ij}x_{ij}, & & & \\
\text{s.t.} & \displaystyle\sum_{i:(i,k)\in A} x_{ik} - \sum_{j:(k,j)\in A} x_{kj} & = & d_k, & k \in N, \\
 & \ell_{ij} \leq x_{ij} & \leq & u_{ij}, & (i,j) \in A.
\end{array}
$$

Minimum cost flows in general networks: LP model and dual

The linear optimization model:

$$\begin{array}{lrcll} \min & \displaystyle\sum_{(i,j)\in A} c_{ij} x_{ij}, & & & \\ \text{s.t.} & \displaystyle\sum_{i:(i,k)\in A} x_{ik} - \sum_{j:(k,j)\in A} x_{kj} & = & d_k, & k \in N, \\ & \ell_{ij} \leq x_{ij} & \leq & u_{ij}, & (i,j) \in A. \end{array}$$

Linear programming dual:

$$\begin{array}{lrcll} \max & \displaystyle\sum_{k\in N} d_k y_k + \sum_{(i,j)\in A} (\ell_{ij}\alpha_{ij} - u_{ij}\beta_{ij}), & & & \\ \text{s.t.} & y_j - y_i + \alpha_{ij} - \beta_{ij} & = & c_{ij}, & (i,j) \in A, \\ & \alpha_{ij}, \beta_{ij} & \geq & 0, & (i,j) \in A. \end{array}$$

The simplex method for minimum cost network flows (Ch. 8.7)

- A solution is optimal if
  - the primal and dual solutions are feasible and
  - the complementary conditions are fulfilled
- Reduced cost: $\overline{c}_{ij} = c_{ij} + y_i - y_j$
- Complementary conditions, $(i, j) \in A$
  - $\alpha_{ij}(x_{ij} - \ell_{ij}) = 0$
  - $\beta_{ij}(u_{ij} - x_{ij}) = 0$
  - $x_{ij}(\overline{c}_{ij} - \alpha_{ij} + \beta_{ij}) = 0$
- Assume that $\ell_{ij} < u_{ij}$.
- A feasible solution x_{ij}, $(i, j) \in A$, is optimal if the following hold:
  - $x_{ij} = u_{ij} \Rightarrow \alpha_{ij} = 0 \Rightarrow$ Reduced cost: $\overline{c}_{ij} = -\beta_{ij} \leq 0$
  - $x_{ij} = \ell_{ij} \Rightarrow \beta_{ij} = 0 \Rightarrow$ Reduced cost: $\overline{c}_{ij} = \alpha_{ij} \geq 0$
  - $\ell_{ij} < x_{ij} < u_{ij} \Rightarrow \alpha_{ij} = \beta_{ij} = 0 \Rightarrow$ Reduced cost: $\overline{c}_{ij} = 0$

The simplex method for minimum cost network flows

- The arc (i, j) corresponds to the variable x_{ij}, $(i, j) \in A$
- A *basic solution* is characterized by the following;
  - If $\ell_{ij} < x_{ij} < u_{ij} \Rightarrow$ the arc (i, j) is in the *basis* $\Leftrightarrow x_{ij}$ is a basic variable
  - If $x_{ij} = \ell_{ij}$ or $x_{ij} = u_{ij} \Rightarrow$ the arc (i, j) *may* be in the *basis* $\Leftrightarrow x_{ij}$ *may be* a basic variable
  - There are exactly $n - 1$ basic arcs which form a *spanning tree* in G (one primal equation is a linear combination of the rest and can thus be removed)

The simplex method for minimum cost flows

1. Find a feasible solution (a spanning tree of basic arcs)
2. Compute reduced costs $\overline{c}_{ij} = c_{ij} + y_i - y_j$ for all non-basic arcs
3. Check termination criteria: If, for every arc (i, j),
   - either: $\overline{c}_{ij} = 0$ and $\ell_{ij} \leq x_{ij} \leq u_{ij}$,
   - or: $\overline{c}_{ij} < 0$ and $x_{ij} = u_{ij}$,
   - or: $\overline{c}_{ij} > 0$ and $x_{ij} = \ell_{ij}$

   hold, then STOP. x_{ij}, $(i, j) \in A$ forms an optimal solution
4. *Entering variable (arc)*: $(p, q) \in \arg\max_{(i,j) \in I} |\overline{c}_{ij}|$
   $I =$ the set of non-basic arcs *not* fulfilling the conditions in 3.
5. *Leaving variable (arc)*: Send flow along the cycle defined by the current *basis* (spanning tree) and the arc (p, q). The arc (i, j) whose flow x_{ij} first reaches u_{ij} or ℓ_{ij} leaves the basis.
6. Go to step 2

- A special case of the network flow model (and of the transportation model)
- Given n persons and n jobs
- Given further the cost c_{ij} of assigning person i to job j
- Binary variables $x_{ij} = 1$ if person i does job j and $x_{ij} = 0$ otherwise
- Find the cheapest assignment of persons to jobs such that all jobs are done

$$\begin{array}{lrcll} \min & \sum_{ij} c_{ij}x_{ij} & & & \\ \text{s.t.} & \sum_j x_{ij} & = & 1 & \forall i \\ & \sum_i x_{ij} & = & 1 & \forall j \\ & x_{ij} & \geq & 0 & \forall i,j \end{array}$$

- The optimal solution is binary (due to the totally unimodular constraint matrix)

An assignment example

- 3 children: John, Karin and Tina
- 3 tasks: mow, paint and wash.
- Given further a "cost" (time, uncomfort,...) for each combination of child/task
- How should the parents distribute the tasks to minimize the cost?

	Mow	Paint	Wash
John	15	10	9
Karin	9	15	10
Tina	10	12	8

- Choose exactly one element in each row and one in each column