MVE165/MMG631
Linear and integer optimization with applications
Lecture 4
Linear programming duality and sensitivity analysis

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A general linear program on "standard form"

- A linear program with n non-negative variables, m equality constraints ($m < n$), and non-negative right hand sides:

$$\begin{aligned} \text{maximize} \quad & z = \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1, \dots, m, \\ & x_j \geq 0, \quad j = 1, \dots, n. \end{aligned}$$

- On matrix form it is expressed as:

$$\begin{aligned} \text{maximize} \quad & z = \mathbf{c}^{\mathrm{T}} \mathbf{x}, \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}^n, \end{aligned}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m_+$ ($\mathbf{b} \geq \mathbf{0}^m$), and $\mathbf{c} \in \mathbb{R}^n$.

An "intuitive" derivation of duality (Ch. 6.1)

- A linear program with optimal value z^* [DRAW GRAPH!!]

$$
\begin{array}{llrrlll}
\text{maximize} & z := & 20x_1 & +18x_2 & & & \text{weights} \\
\text{subject to} & & 7x_1 & +10x_2 & \leq 3600 & (1) & v_1 \\
 & & 16x_1 & +12x_2 & \leq 5400 & (2) & v_2 \\
 & & & x_1, x_2 & \geq 0 & &
\end{array}
$$

- How large can z^* be?
- Compute upper estimates of z^*, e.g.
  - Multiply (1) by 3 $\Rightarrow$ $21x_1 + 30x_2 \leq 10800$ $\Rightarrow$ $z^* \leq 10800$
  - Multiply (2) by 1.5 $\Rightarrow$ $24x_1 + 18x_2 \leq 8100$ $\Rightarrow$ $z^* \leq 8100$
  - Combine: $0.6\times(1)+1\times(2) \Rightarrow 20.2x_1+18x_2 \leq 7560 \Rightarrow z^* \leq 7560$
- Do better than guess—compute *optimal* weights!
- Value of estimate: $w = 3600v_1 + 5400v_2 \rightarrow \min$
- Constraints on weights: $\left[\begin{array}{rl} 7v_1 + 16v_2 & \geq 20 \\ 10v_1 + 12v_2 & \geq 18 \\ v_1, v_2 & \geq 0 \end{array}\right]$

The best (lowest) possible upper estimate of z^*

$$
\begin{array}{lrcrl}
\text{minimize} & w := 3600v_1 & + & 5400v_2 & \\
\text{subject to} & 7v_1 & + & 16v_2 & \geq 20 \\
& 10v_1 & + & 12v_2 & \geq 18 \\
& & & v_1, v_2 & \geq 0
\end{array}
$$

- A linear program! [Draw graph!!]
- It is called the linear programming **dual** of the original linear program

The lego model – the market problem

- Consider the lego problem

$$\begin{array}{llcrcrcl} \text{maximize} & z & = & 1600x_1 & + & 1000x_2 & & \\ \text{subject to} & & & 2x_1 & + & x_2 & \leq & 6 \\ & & & 2x_1 & + & 2x_2 & \leq & 8 \\ & & & & & x_1,\ x_2 & \geq & 0 \end{array}$$

- Option: Sell bricks instead of making furniture
- $v_1 (v_2)$ = price of a large (small) brick
- Market wish to minimize payment: *minimize* $6v_1 + 8v_2$
- I sell if prices are high enough:
  - $2v_1 + 2v_2 \geq 1600$ – otherwise better to make tables
  - $v_1 + 2v_2 \geq 1000$ – otherwise better to make chairs
  - $v_1, v_2 \geq 0$ – prices are naturally non-negative

Linear programming duality

- To each primal linear program corresponds a dual linear program

$$
\begin{aligned}
\text{[Primal]} \quad & \text{minimize} & z &= \mathbf{c}^{\mathrm{T}}\mathbf{x}, \\
& \text{subject to} & \mathbf{A}\mathbf{x} &= \mathbf{b}, \\
& & \mathbf{x} &\geq \mathbf{0}^n,
\end{aligned}
$$

$$
\begin{aligned}
\text{[Dual]} \quad & \text{maximize} & w &= \mathbf{b}^{\mathrm{T}}\mathbf{y}, \\
& \text{subject to} & \mathbf{A}^{\mathrm{T}}\mathbf{y} &\leq \mathbf{c}.
\end{aligned}
$$

- On component form:

$$
\begin{array}{llllll}
\text{[Primal]} & \text{minimize} & z = \sum_{j=1}^{n} c_j x_j & & & \\
& \text{subject to} & \sum_{j=1}^{n} a_{ij} x_j & = & b_i, & i = 1, \ldots, m, \\
& & x_j & \geq & 0, & j = 1, \ldots, n, \\
\hline
\text{[Dual]} & \text{maximize} & w = \sum_{j=1}^{n} b_i y_i & & & \\
& \text{subject to} & \sum_{i=1}^{m} a_{ij} y_i & \leq & c_j, & j = 1, \ldots, n.
\end{array}
$$

An example of linear programming duality

- A primal linear program

$$
\begin{array}{rrrrl}
\text{minimize} & z = & 2x_1 & +3x_2 & \\
\text{subject to} & & 3x_1 & +2x_2 & = 14 \\
 & & 2x_1 & -4x_2 & \geq 2 \\
 & & 4x_1 & +3x_2 & \leq 19 \\
 & & & x_1, x_2 & \geq 0
\end{array}
$$

- The corresponding dual linear program

$$
\begin{array}{rrrrrl}
\text{maximize} & w = & 14y_1 & +2y_2 & +19y_3 & \\
\text{subject to} & & 3y_1 & +2y_2 & +4y_3 & \leq 2 \\
 & & 2y_1 & -4y_2 & +3y_3 & \leq 3 \\
 & & y_1 & & & \text{free}, \\
 & & & y_2 & & \geq 0, \\
 & & & & y_3 & \leq 0
\end{array}
$$

Rules for constructing the dual program (Ch. 6.2)

maximization	$\Leftrightarrow$	minimization
dual program	$\Leftrightarrow$	primal program
primal program	$\Leftrightarrow$	dual program
constraints		*variables*
$\geq$	$\Leftrightarrow$	≤ 0
$\leq$	$\Leftrightarrow$	≥ 0
$=$	$\Leftrightarrow$	free
variables		*constraints*
≥ 0	$\Leftrightarrow$	$\geq$
≤ 0	$\Leftrightarrow$	$\leq$
free	$\Leftrightarrow$	$=$

The dual of the dual of any linear program equals the primal

- **Weak duality** [Th. 6.1]:
Let $\mathbf{x}$ be a feasible point in the primal (minimization) and $\mathbf{y}$ be a feasible point in the dual (maximization). Then,

$$z = \mathbf{c}^{\mathrm{T}}\mathbf{x} \geq \mathbf{b}^{\mathrm{T}}\mathbf{y} = w$$

- **Strong duality** [Th. 6.3]:
In a pair of primal and dual linear programs, if one of them has an optimal solution, so does the other, and their optimal values are equal.

- **Complementary slackness** [Th. 6.5]:
If $\mathbf{x}$ is optimal in the primal and $\mathbf{y}$ is optimal in the dual,
$\Rightarrow$ then $\mathbf{x}^{\mathrm{T}}(\mathbf{c} - \mathbf{A}^{\mathrm{T}}\mathbf{y}) = \mathbf{y}^{\mathrm{T}}(\mathbf{b} - \mathbf{A}\mathbf{x}) = 0$.
If $\mathbf{x}$ is feasible in the primal, $\mathbf{y}$ is feasible in the dual, and $\mathbf{x}^{\mathrm{T}}(\mathbf{c} - \mathbf{A}^{\mathrm{T}}\mathbf{y}) = \mathbf{y}^{\mathrm{T}}(\mathbf{b} - \mathbf{A}\mathbf{x}) = 0$,
$\Rightarrow$ then $\mathbf{x}$ and $\mathbf{y}$ are optimal for their respective problems.

Relations between primal and dual optimal solutions

primal (dual) problem		dual (primal) problem
unique and non-degenerate solution	$\iff$	unique and non-degenerate solution
unbounded solution	$\implies$	no feasible solutions
no feasible solutions	$\implies$	unbounded solution **or** no feasible solutions
degenerate solution	$\iff$	alternative solutions

Exercises on duality

HOMEWORK!

- Formulate and solve graphically the dual of:

$$\begin{array}{rrrrrl} \text{minimize} & z = & 6x_1 & +3x_2 & +x_3 & \\ \text{subject to} & & 6x_1 & -3x_2 & +x_3 & \geq 2 \\ & & 3x_1 & +4x_2 & +x_3 & \geq 5 \\ & & & & x_1, x_2, x_3 & \geq 0 \end{array}$$

- Then find the optimal primal solution

- Verify that the dual of the dual equals the primal

Derivation of the simplex method (repetition) (Ch. 4.8)

- B = set of basic variables, N = set of non-basic variables

$\Rightarrow$ $|B| = m$ and $|N| = n - m$

- Partition matrix/vectors: $\mathbf{A} = (\mathbf{B}, \mathbf{N})$, $\mathbf{x} = (\mathbf{x}_B, \mathbf{x}_N)$, $\mathbf{c} = (\mathbf{c}_B, \mathbf{c}_N)$
- The matrix $\mathbf{B}$ ($\mathbf{N}$) contains the columns of $\mathbf{A}$ corresponding to the index set B (N) — Analogously for $\mathbf{x}$ and $\mathbf{c}$
- Rewrite the linear program:

$$\begin{bmatrix} \text{minimize } z = \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to} \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \\ \mathbf{x} \geq \mathbf{0}^n \end{bmatrix} = \begin{bmatrix} \text{minimize } z = \mathbf{c}_B^{\mathrm{T}}\mathbf{x}_B + \mathbf{c}_N^{\mathrm{T}}\mathbf{x}_N \\ \text{subject to} \quad \mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_N = \mathbf{b}, \\ \mathbf{x}_B \geq \mathbf{0}^m, \; \mathbf{x}_N \geq \mathbf{0}^{n-m} \end{bmatrix}$$

- Substitute: $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N \Longrightarrow$

$$\begin{aligned} \text{minimize} \quad & z = \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{b} + [\mathbf{c}_N^{\mathrm{T}} - \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{N}]\mathbf{x}_N \\ \text{subject to} \quad & \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N \geq \mathbf{0}^m, \\ & \mathbf{x}_N \geq \mathbf{0}^{n-m} \end{aligned}$$

Optimality and feasibility (repetition)

- **Optimality condition** (for minimization)

  The basis B is optimal if $\mathbf{c}_N^{\mathrm{T}} - \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{N} \geq \mathbf{0}^{n-m}$
  (marginal values = reduced costs ≥ 0)

  If not, choose as entering variable $j \in N$ the one with the largest negative value of the reduced cost $c_j - \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{A}_j$

- **Feasibility condition**

  For all $i \in B$ it holds that $x_i = (\mathbf{B}^{-1}\mathbf{b})_i - (\mathbf{B}^{-1}\mathbf{A}_j)_i x_j$

  Choose the leaving variable $i^* \in B$ according to

$$i^* = \arg\min_{i \in B} \left\{ \frac{(\mathbf{B}^{-1}\mathbf{b})_i}{(\mathbf{B}^{-1}\mathbf{A}_j)_i} \,\middle|\, (\mathbf{B}^{-1}\mathbf{A}_j)_i > 0 \right\}$$

In the simplex tableau, we have

basis	$-z$	$\mathbf{x}_B$	$\mathbf{x}_N$	$\mathbf{s}$	RHS
$-z$	1	$\mathbf{0}$	$\mathbf{c}_N^{\mathrm{T}} - \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{N}$	$-\mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}$	$-\mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{b}$
$\mathbf{x}_B$	$\mathbf{0}$	$\mathbf{I}$	$\mathbf{B}^{-1}\mathbf{N}$	$\mathbf{B}^{-1}$	$\mathbf{B}^{-1}\mathbf{b}$

- $\mathbf{s}$ denotes possible slack variables (columns for $\mathbf{s}$ are copies of certain columns for $(\mathbf{x}_B, \mathbf{x}_N)$)
- The computations performed by the simplex algorithm involve matrix inversions and updates of these
- A non-basic (basic) variable enters (leaves) the basis $\Rightarrow$ one column, $\mathbf{A}_j$, of $\mathbf{B}$ is replaced by another, $\mathbf{A}_k$
- Row operations $\Leftrightarrow$ Updates of $\mathbf{B}^{-1}$ (and $\mathbf{B}^{-1}\mathbf{N}$, $\mathbf{B}^{-1}\mathbf{b}$, and $\mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}$)

$\Rightarrow$ Efficient numerical computations are crucial for the performance of the simplex algorithm

- How does the optimum change when the right hand sides (resources, e.g.) change?
- When the objective coefficients (prices, e.g.) change?
- Assume that the basis B is optimal:

$$\begin{aligned} \text{minimize} \quad & z = \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{b} + [\mathbf{c}_N^{\mathrm{T}} - \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{N}]\mathbf{x}_N \\ \text{subject to} \quad & \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N \geq \mathbf{0}^m, \\ & \mathbf{x}_N \geq \mathbf{0}^{n-m} \end{aligned}$$

- $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N$

Changes in the right hand side coefficients

- The *shadow price* [Def. 5.3] of a constraint is defined as the change in the optimal value as a function of the (marginal) change in the RHS. It equals the optimal value of the corresponding dual variable.
- Suppose $\mathbf{b}$ changes to $\mathbf{b} + \Delta\mathbf{b}$

$\Rightarrow$ New optimal value:

$$z^{\text{new}} = \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}(\mathbf{b} + \Delta\mathbf{b}) = z + \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\Delta\mathbf{b}$$

- The current basis is feasible if $\mathbf{B}^{-1}(\mathbf{b} + \Delta\mathbf{b}) \geq 0$
- If not: negative values will occur in the RHS of the simplex tableau
- The reduced costs are unchanged (positive, at optimum) $\Rightarrow$ this can be resolved using the *dual simplex method*

Changes in the right hand side coefficients

- Consider the linear program

$$\begin{array}{rrrrl} \text{minimize} & z = & -x_1 & -2x_2 & \\ \text{subject to} & & -2x_1 & +x_2 & \leq 2 \\ & & -x_1 & +2x_2 & \leq 7 \\ & & x_1 & & \leq 3 \\ & & & x_1, x_2 & \geq 0 \end{array}$$

DRAW GRAPH!!

- The optimal solution is given by

basis	$-z$	x_1	x_2	s_1	s_2	s_3	RHS
$-z$	1	0	0	0	1	2	13
x_2	0	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	5
x_1	0	1	0	0	0	1	3
s_1	0	0	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	3

Changes in the right hand side coefficients

- Change the right hand side according to

$$
\begin{array}{lrrrl}
\text{minimize} & z = & -x_1 & -2x_2 & \\
\text{subject to} & & -2x_1 & +x_2 & \leq 2 \\
& & -x_1 & +2x_2 & \leq 7+\delta \\
& & x_1 & & \leq 3 \\
& & x_1, x_2 & \geq 0 &
\end{array}
$$

- The change in the right hand side is given by $\mathbf{B}^{-1}(0, \delta, 0)^{\mathrm{T}} = (\frac{1}{2}\delta, 0, -\frac{1}{2}\delta)^{\mathrm{T}} \Rightarrow$ new optimal tableau:

basis	$-z$	x_1	x_2	s_1	s_2	s_3	RHS
$-z$	1	0	0	0	1	2	$13+\delta$
x_2	0	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	$5+\frac{1}{2}\delta$
x_1	0	1	0	0	0	1	3
s_1	0	0	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	$3-\frac{1}{2}\delta$

- The current basis is feasible if $-10 \leq \delta \leq 6$

Changes in the right hand side coefficients

- Suppose $\delta = 8$:

basis	$-z$	x_1	x_2	s_1	s_2	s_3	RHS
$-z$	1	0	0	0	1	2	21
x_2	0	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	9
x_1	0	1	0	0	0	1	3
s_1	0	0	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	-1

- Dual simplex iteration:
- $s_1 = -1$ has to leave the basis
- Find the smallest ratio between reduced costs (for non-basic columns) and (negative) elements in the "s_1-row" (to stay optimal)
- s_2 will enter the basis — **New optimal** tableau:

basis	$-z$	x_1	x_2	s_1	s_2	s_3	RHS
$-z$	1	0	0	2	0	5	19
x_2	0	0	1	1	0	2	8
x_1	0	1	0	0	0	1	3
s_2	0	0	0	-2	1	-3	2

Changes in the objective coefficients

- The *reduced cost* of a non-basic variable defines the change in the objective value when the value of the corresponding variable is (marginally) increased.
  The basis B is optimal if $\mathbf{c}_N^{\mathrm{T}} - \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{N} \geq \mathbf{0}^{n-m}$ (marginal values = reduced costs ≥ 0)
- Suppose $\mathbf{c}$ changes to $\mathbf{c} + \Delta\mathbf{c}$
- The new optimal value:

$$z^{\mathrm{new}} = (\mathbf{c}_B + \Delta\mathbf{c}_B)^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{b} = z + \Delta\mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{b}$$

- The current basis is optimal if
  $(\mathbf{c}_N + \Delta\mathbf{c}_N)^{\mathrm{T}} - (\mathbf{c}_B + \Delta\mathbf{c}_B)^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{N} \geq \mathbf{0}$
- If not: more simplex iterations to find the optimal solution

Changes in the objective coefficients

- Change the objective according to

$$\begin{array}{lrrrl} \text{minimize} & z = & -x_1 & +(-2+\delta)x_2 & \\ \text{subject to} & & -2x_1 & +x_2 & \leq 2 \\ & & -x_1 & +2x_2 & \leq 7 \\ & & x_1 & & \leq 3 \\ & & x_1, x_2 & \geq 0 & \end{array}$$

- The changes in the reduced costs are given by $-(\delta, 0, 0)\mathbf{B}^{-1}\mathbf{N} = (-\frac{1}{2}\delta, -\frac{1}{2}\delta) \Rightarrow$ new optimal tableau:

basis	$-z$	x_1	x_2	s_1	s_2	s_3	RHS
$-z$	1	0	0	0	$1-\frac{1}{2}\delta$	$2-\frac{1}{2}\delta$	$13-5\delta$
x_2	0	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	5
x_1	0	1	0	0	0	1	3
s_1	0	0	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	3

- The current basis is optimal if $\delta \leq 2$

Changes in the objective coefficients

- Suppose $\delta = 4$: new tableau:

basis	$-z$	x_1	x_2	s_1	s_2	s_3	RHS
$-z$	1	0	0	0	-1	0	-7
x_2	0	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	5
x_1	0	1	0	0	0	1	3
s_1	0	0	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	3

- Let s_2 enter and x_2 leave the basis. New optimal tableau:

basis	$-z$	x_1	x_2	s_1	s_2	s_3	RHS
$-z$	1	0	2	0	0	1	3
s_2	0	0	2	0	1	1	10
x_1	0	1	0	0	0	1	3
s_1	0	0	1	1	0	2	8