MVE165/MMG631
Linear and integer optimization with applications
Lecture 4
Linear programming duality and sensitivity analysis

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A general linear program on "standard form"

- A linear program with n non-negative variables, m equality constraints ($m < n$), and non-negative right hand sides:

$$\text{maximize} \quad z = \sum_{j=1}^{n} c_j x_j$$
$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1, \ldots, m,$$
$$x_j \geq 0, \quad j = 1, \ldots, n.$$

- On matrix form it is expressed as:

$$\text{maximize} \quad z = \mathbf{c}^{\mathrm{T}} \mathbf{x},$$
$$\text{subject to} \quad \mathbf{A}\mathbf{x} = \mathbf{b},$$
$$\mathbf{x} \geq \mathbf{0}^n,$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m_+$ ($\mathbf{b} \geq \mathbf{0}^m$), and $\mathbf{c} \in \mathbb{R}^n$.

An "intuitive" derivation of duality (Ch. 6.1)

- A linear program with optimal value z^* [DRAW GRAPH!!]

maximize	$z :=$	$20x_1$	$+18x_2$			weights
subject to		$7x_1$	$+10x_2$	≤ 3600	(1)	v_1
		$16x_1$	$+12x_2$	≤ 5400	(2)	v_2
			x_1, x_2	≥ 0		

- How large can z^* be?
- Compute upper estimates of z^*, e.g.
  - Multiply (1) by 3 $\Rightarrow 21x_1 + 30x_2 \leq 10800 \Rightarrow z^* \leq 10800$
  - Multiply (2) by 1.5 $\Rightarrow 24x_1 + 18x_2 \leq 8100 \Rightarrow z^* \leq 8100$
  - Combine: $0.6\times(1)+1\times(2) \Rightarrow 20.2x_1+18x_2 \leq 7560 \Rightarrow z^* \leq 7560$
- Do better than guess—compute *optimal* weights!
- Value of estimate: $w = 3600v_1 + 5400v_2 \rightarrow \min$
- Constraints on weights: $\begin{bmatrix} 7v_1 + 16v_2 & \geq 20 \\ 10v_1 + 12v_2 & \geq 18 \\ v_1, v_2 & \geq 0 \end{bmatrix}$

The best (lowest) possible upper estimate of z^*

$$
\begin{array}{rrcrcl}
\text{minimize} & w := & 3600v_1 & + & 5400v_2 & \\
\text{subject to} & & 7v_1 & + & 16v_2 & \geq 20 \\
& & 10v_1 & + & 12v_2 & \geq 18 \\
& & & & v_1, v_2 & \geq 0
\end{array}
$$

- A linear program!   [DRAW GRAPH!!]

- It is called the linear programming **dual** of the original linear program

The lego model – the market problem

▶ Consider the lego problem

$$
\begin{array}{rrcrcl}
\text{maximize} & z \;=\; 1600x_1 & + & 1000x_2 & & \\
\text{subject to} & 2x_1 & + & x_2 & \leq & 6 \\
& 2x_1 & + & 2x_2 & \leq & 8 \\
& & & x_1, \; x_2 & \geq & 0
\end{array}
$$

▶ Option: Sell bricks instead of making furniture

▶ $v_1 (v_2) =$ price of a large (small) brick

▶ Market wish to minimize payment: *minimize* $\; 6v_1 + 8v_2$

▶ I sell if prices are high enough:
  ▶ $2v_1 + 2v_2 \geq 1600$ – otherwise better to make tables
  ▶ $v_1 + 2v_2 \geq 1000$ – otherwise better to make chairs
  ▶ $v_1, v_2 \geq 0$ – prices are naturally non-negative

Linear programming duality

▶ To each primal linear program corresponds a dual linear program

[Primal] minimize $\quad z = \mathbf{c}^{\mathrm{T}}\mathbf{x}$,

subject to $\quad \mathbf{A}\mathbf{x} = \mathbf{b}$,

$\quad\quad\quad\quad \mathbf{x} \geq \mathbf{0}^n$,

[Dual] maximize $\quad w = \mathbf{b}^{\mathrm{T}}\mathbf{y}$,

subject to $\quad \mathbf{A}^{\mathrm{T}}\mathbf{y} \leq \mathbf{c}$.

▶ On component form:

[Primal]	minimize	$z = \sum_{j=1}^{n} c_j x_j$			
	subject to	$\sum_{j=1}^{n} a_{ij} x_j$	$=$	$b_i,$	$i = 1, \ldots, m,$
		x_j	$\geq$	$0,$	$j = 1, \ldots, n,$
[Dual]	maximize	$w = \sum_{i=1}^{m} b_i y_i$			
	subject to	$\sum_{i=1}^{m} a_{ij} y_i$	$\leq$	$c_j,$	$j = 1, \ldots, n.$

An example of linear programming duality

- A primal linear program

$$
\begin{array}{rlrl}
\text{minimize} & z = & 2x_1 & +3x_2 \\
\text{subject to} & & 3x_1 & +2x_2 & = 14 \\
& & 2x_1 & -4x_2 & \geq 2 \\
& & 4x_1 & +3x_2 & \leq 19 \\
& & & x_1, x_2 & \geq 0
\end{array}
$$

- The corresponding dual linear program

$$
\begin{array}{rlrll}
\text{maximize} & w = & 14y_1 & +2y_2 & +19y_3 \\
\text{subject to} & & 3y_1 & +2y_2 & +4y_3 & \leq 2 \\
& & 2y_1 & -4y_2 & +3y_3 & \leq 3 \\
& & y_1 & & & \text{free,} \\
& & & y_2 & & \geq 0, \\
& & & & y_3 & \leq 0
\end{array}
$$

maximization	$\Leftrightarrow$	minimization
dual program	$\Leftrightarrow$	primal program
primal program	$\Leftrightarrow$	dual program
constraints		*variables*
$\geq$	$\Leftrightarrow$	≤ 0
$\leq$	$\Leftrightarrow$	≥ 0
$=$	$\Leftrightarrow$	free
variables		*constraints*
≥ 0	$\Leftrightarrow$	$\geq$
≤ 0	$\Leftrightarrow$	$\leq$
free	$\Leftrightarrow$	$=$

The dual of the dual of any linear program equals the primal

► **Weak duality** [Th. 6.1]:
Let **x** be a feasible point in the primal (minimization) and **y** be a feasible point in the dual (maximization). Then,

$$z = \mathbf{c}^{\mathrm{T}}\mathbf{x} \geq \mathbf{b}^{\mathrm{T}}\mathbf{y} = w$$

► **Strong duality** [Th. 6.3]:
In a pair of primal and dual linear programs, if one of them has an optimal solution, so does the other, and their optimal values are equal.

► **Complementary slackness** [Th. 6.5]:
If **x** is optimal in the primal and **y** is optimal in the dual,
⇒ then $\mathbf{x}^{\mathrm{T}}(\mathbf{c} - \mathbf{A}^{\mathrm{T}}\mathbf{y}) = \mathbf{y}^{\mathrm{T}}(\mathbf{b} - \mathbf{A}\mathbf{x}) = 0$.
If **x** is feasible in the primal, **y** is feasible in the dual,
and $\mathbf{x}^{\mathrm{T}}(\mathbf{c} - \mathbf{A}^{\mathrm{T}}\mathbf{y}) = \mathbf{y}^{\mathrm{T}}(\mathbf{b} - \mathbf{A}\mathbf{x}) = 0$,
⇒ then **x** and **y** are optimal for their respective problems.

Relations between primal and dual optimal solutions

primal (dual) problem	$\iff$	dual (primal) problem
unique and non-degenerate solution	$\iff$	unique and non-degenerate solution
unbounded solution	$\implies$	no feasible solutions
no feasible solutions	$\implies$	unbounded solution **or** no feasible solutions
degenerate solution	$\iff$	alternative solutions

Exercises on duality

HOMEWORK!

- Formulate and solve graphically the dual of:

$$\begin{array}{rllll}
\text{minimize} & z = & 6x_1 & +3x_2 & +x_3 \\
\text{subject to} & & 6x_1 & -3x_2 & +x_3 & \geq 2 \\
& & 3x_1 & +4x_2 & +x_3 & \geq 5 \\
& & & & x_1, x_2, x_3 & \geq 0
\end{array}$$

- Then find the optimal primal solution

- Verify that the dual of the dual equals the primal

Derivation of the simplex method (repetition)   (Ch. 4.8)

- B = set of basic variables, N = set of non-basic variables
- $\Rightarrow |B| = m$ and $|N| = n - m$
- Partition matrix/vectors: $\mathbf{A} = (\mathbf{B}, \mathbf{N})$, $\mathbf{x} = (\mathbf{x}_B, \mathbf{x}_N)$, $\mathbf{c} = (\mathbf{c}_B, \mathbf{c}_N)$
- The matrix $\mathbf{B}$ ($\mathbf{N}$) contains the columns of $\mathbf{A}$ corresponding to the index set B (N) — Analogously for $\mathbf{x}$ and $\mathbf{c}$
- Rewrite the linear program:

$$\begin{bmatrix} \text{minimize } z = \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to} \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \\ \mathbf{x} \geq \mathbf{0}^n \end{bmatrix} = \begin{bmatrix} \text{minimize } z = \mathbf{c}_B^{\mathrm{T}}\mathbf{x}_B + \mathbf{c}_N^{\mathrm{T}}\mathbf{x}_N \\ \text{subject to} \quad \mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_N = \mathbf{b}, \\ \mathbf{x}_B \geq \mathbf{0}^m, \ \mathbf{x}_N \geq \mathbf{0}^{n-m} \end{bmatrix}$$

- Substitute: $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N \Longrightarrow$

$$\begin{aligned} \text{minimize} \quad & z = \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{b} + [\mathbf{c}_N^{\mathrm{T}} - \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{N}]\mathbf{x}_N \\ \text{subject to} \quad & \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N \geq \mathbf{0}^m, \\ & \mathbf{x}_N \geq \mathbf{0}^{n-m} \end{aligned}$$

Optimality and feasibility (repetition)

▶ **Optimality condition** (for minimization)

The basis B is optimal if $\mathbf{c}_N^{\mathrm{T}} - \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{N} \geq \mathbf{0}^{n-m}$
(marginal values = reduced costs ≥ 0)

If not, choose as entering variable $j \in N$ the one with the largest negative value of the reduced cost $c_j - \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{A}_j$

▶ **Feasibility condition**

For all $i \in B$ it holds that $x_i = (\mathbf{B}^{-1}\mathbf{b})_i - (\mathbf{B}^{-1}\mathbf{A}_j)_i x_j$

Choose the leaving variable $i^* \in B$ according to

$$i^* = \arg\min_{i \in B}\left\{ \left. \frac{(\mathbf{B}^{-1}\mathbf{b})_i}{(\mathbf{B}^{-1}\mathbf{A}_j)_i} \right| (\mathbf{B}^{-1}\mathbf{A}_j)_i > 0 \right\}$$

In the simplex tableau, we have

basis	$-z$	$\mathbf{x}_B$	$\mathbf{x}_N$	$\mathbf{s}$	RHS
$-z$	1	$\mathbf{0}$	$\mathbf{c}_N^{\mathrm{T}} - \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{N}$	$-\mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}$	$-\mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{b}$
$\mathbf{x}_B$	$\mathbf{0}$	$\mathbf{I}$	$\mathbf{B}^{-1}\mathbf{N}$	$\mathbf{B}^{-1}$	$\mathbf{B}^{-1}\mathbf{b}$

- ▶ $\mathbf{s}$ denotes possible slack variables (columns for $\mathbf{s}$ are copies of certain columns for $(\mathbf{x}_B, \mathbf{x}_N)$)
- ▶ The computations performed by the simplex algorithm involve matrix inversions and updates of these
- ▶ A non-basic (basic) variable enters (leaves) the basis $\Rightarrow$ one column, $\mathbf{A}_j$, of $\mathbf{B}$ is replaced by another, $\mathbf{A}_k$
- ▶ Row operations $\Leftrightarrow$ Updates of $\mathbf{B}^{-1}$ (and $\mathbf{B}^{-1}\mathbf{N}$, $\mathbf{B}^{-1}\mathbf{b}$, and $\mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}$)
- $\Rightarrow$ Efficient numerical computations are crucial for the performance of the simplex algorithm

- How does the optimum change when the right hand sides (resources, e.g.) change?

- When the objective coefficients (prices, e.g.) change?

- Assume that the basis B is optimal:

$$\begin{aligned}
\text{minimize} \quad & z = \mathbf{c}_B^{\mathrm{T}} \mathbf{B}^{-1} \mathbf{b} + [\mathbf{c}_N^{\mathrm{T}} - \mathbf{c}_B^{\mathrm{T}} \mathbf{B}^{-1} \mathbf{N}] \mathbf{x}_N \\
\text{subject to} \quad & \mathbf{B}^{-1} \mathbf{b} - \mathbf{B}^{-1} \mathbf{N} \mathbf{x}_N \geq \mathbf{0}^m, \\
& \mathbf{x}_N \geq \mathbf{0}^{n-m}
\end{aligned}$$

- $\mathbf{x}_B = \mathbf{B}^{-1} \mathbf{b} - \mathbf{B}^{-1} \mathbf{N} \mathbf{x}_N$

Changes in the right hand side coefficients

- The *shadow price* [Def. 5.3] of a constraint is defined as the change in the optimal value as a function of the (marginal) change in the RHS. It equals the optimal value of the corresponding dual variable.

- Suppose $\mathbf{b}$ changes to $\mathbf{b} + \Delta\mathbf{b}$

$\Rightarrow$ New optimal value:

$$z^{\mathrm{new}} = \mathbf{c}_B^{\mathrm{T}} \mathbf{B}^{-1}(\mathbf{b} + \Delta\mathbf{b}) = z + \mathbf{c}_B^{\mathrm{T}} \mathbf{B}^{-1} \Delta\mathbf{b}$$

- The current basis is feasible if $\mathbf{B}^{-1}(\mathbf{b} + \Delta\mathbf{b}) \geq 0$

- If not: negative values will occur in the RHS of the simplex tableau

- The reduced costs are unchanged (positive, at optimum)
  $\Rightarrow$ this can be resolved using the *dual simplex method*

Changes in the right hand side coefficients

- Consider the linear program

$$
\begin{array}{rrrl}
\text{minimize} \quad z = & -x_1 & -2x_2 & \\
\text{subject to} & -2x_1 & +x_2 & \leq 2 \\
& -x_1 & +2x_2 & \leq 7 \\
\text{DRAW GRAPH!!} & x_1 & & \leq 3 \\
& x_1, x_2 & & \geq 0
\end{array}
$$

- The optimal solution is given by

basis	$-z$	x_1	x_2	s_1	s_2	s_3	RHS
$-z$	1	0	0	0	1	2	13
x_2	0	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	5
x_1	0	1	0	0	0	1	3
s_1	0	0	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	3

Changes in the right hand side coefficients

- Change the right hand side according to

$$\begin{aligned}
\text{minimize} \quad z = \quad & -x_1 \quad -2x_2 \\
\text{subject to} \quad & -2x_1 \quad +x_2 \quad \leq 2 \\
& -x_1 \quad +2x_2 \quad \leq 7 + \delta \\
& x_1 \qquad\qquad \leq 3 \\
& x_1, x_2 \quad \geq 0
\end{aligned}$$

- The change in the right hand side is given by
  $\mathbf{B}^{-1}(0, \delta, 0)^{\mathrm{T}} = (\frac{1}{2}\delta, 0, -\frac{1}{2}\delta)^{\mathrm{T}} \Rightarrow$ new optimal tableau:

basis	$-z$	x_1	x_2	s_1	s_2	s_3	RHS
$-z$	1	0	0	0	1	2	$13 + \delta$
x_2	0	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	$5 + \frac{1}{2}\delta$
x_1	0	1	0	0	0	1	3
s_1	0	0	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	$3 - \frac{1}{2}\delta$

- The current basis is feasible if $-10 \leq \delta \leq 6$

Changes in the right hand side coefficients

- Suppose $\delta = 8$:

basis	$-z$	x_1	x_2	s_1	s_2	s_3	RHS
$-z$	1	0	0	0	1	2	21
x_2	0	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	9
x_1	0	1	0	0	0	1	3
s_1	0	0	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	-1

- Dual simplex iteration:
- $s_1 = -1$ has to leave the basis
- Find the smallest ratio between reduced costs (for non-basic columns) and (negative) elements in the "s_1-row" (to stay optimal)
- s_2 will enter the basis — **New optimal** tableau:

basis	$-z$	x_1	x_2	s_1	s_2	s_3	RHS
$-z$	1	0	0	2	0	5	19
x_2	0	0	1	1	0	2	8
x_1	0	1	0	0	0	1	3
s_2	0	0	0	-2	1	-3	2

Changes in the objective coefficients

- The *reduced cost* of a non-basic variable defines the change in the objective value when the value of the corresponding variable is (marginally) increased.
  The basis B is optimal if $\mathbf{c}_N^{\mathrm{T}} - \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{N} \geq \mathbf{0}^{n-m}$ (marginal values = reduced costs ≥ 0)

- Suppose $\mathbf{c}$ changes to $\mathbf{c} + \Delta\mathbf{c}$

- The new optimal value:

$$z^{\mathrm{new}} = (\mathbf{c}_B + \Delta\mathbf{c}_B)^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{b} = z + \Delta\mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{b}$$

- The current basis is optimal if
  $(\mathbf{c}_N + \Delta\mathbf{c}_N)^{\mathrm{T}} - (\mathbf{c}_B + \Delta\mathbf{c}_B)^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{N} \geq \mathbf{0}$

- If not: more simplex iterations to find the optimal solution

Changes in the objective coefficients

▶ Change the objective according to

$$
\begin{array}{rrll}
\text{minimize} \quad z = & -x_1 & +(-2+\delta)x_2 & \\
\text{subject to} & -2x_1 & +x_2 & \leq 2 \\
& -x_1 & +2x_2 & \leq 7 \\
& x_1 & & \leq 3 \\
& x_1, x_2 & & \geq 0
\end{array}
$$

▶ The changes in the reduced costs are given by
$-(\delta, 0, 0)\mathbf{B}^{-1}\mathbf{N} = (-\frac{1}{2}\delta, -\frac{1}{2}\delta) \Rightarrow$ new optimal tableau:

basis	$-z$	x_1	x_2	s_1	s_2	s_3	RHS
$-z$	1	0	0	0	$1-\frac{1}{2}\delta$	$2-\frac{1}{2}\delta$	$13-5\delta$
x_2	0	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	5
x_1	0	1	0	0	0	1	3
s_1	0	0	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	3

▶ The current basis is optimal if $\delta \leq 2$

Changes in the objective coefficients

- Suppose $\delta = 4$: new tableau:

basis	$-z$	x_1	x_2	s_1	s_2	s_3	RHS
$-z$	1	0	0	0	-1	0	-7
x_2	0	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	5
x_1	0	1	0	0	0	1	3
s_1	0	0	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	3

- Let s_2 enter and x_2 leave the basis. New optimal tableau:

basis	$-z$	x_1	x_2	s_1	s_2	s_3	RHS
$-z$	1	0	2	0	0	1	3
s_2	0	0	2	0	1	1	10
x_1	0	1	0	0	0	1	3
s_1	0	0	1	1	0	2	8