MVE165/MMG631
Linear and integer optimization with applications
Lecture 6a
Theory and algorithms for discrete optimization models

Ann-Brith Strömberg

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Methods for ILP: Overview (Ch. 14.1)

- Enumeration
  - Implicit enumeration: Branch–and–bound
- Relaxations
  - Decomposition methods: Solve simpler problems repeatedly
  - Add valid inequalities to an LP – "cutting plane methods"
  - Lagrangian relaxation
- Heuristic algorithms – optimum *not* guaranteed
  - "Simple" rules $\Rightarrow$ feasible solutions
  - Construction heuristics
  - Local search heuristics

Relaxations and feasible solutions (Ch. 14.2)

- Consider a minimization integer linear program (ILP):

$$
\text{[ILP]} \qquad z^* := \begin{array}{rrcll} \min & \mathbf{c}^{\mathrm{T}}\mathbf{x} & & & \\ \text{subject to} & \mathbf{A}\mathbf{x} & \leq & \mathbf{b} & \\ & \mathbf{x} & \geq & \mathbf{0} & \text{and integer} \end{array}
$$

- The feasible set $X = \{\mathbf{x} \in Z_+^n \,|\, \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$ is *non*-convex
- How prove that a solution $\mathbf{x}^* \in X$ is optimal?
- We cannot use strong duality/complementarity as for linear optimization (where X is polyhedral $\Rightarrow$ convexity)
- Bounds on the optimal value
  - Optimistic estimate $\underline{z} \leq z^*$ from a *relaxation* of ILP
  - Pessimistic estimate $\bar{z} \geq z^*$ from a *feasible solution* to ILP
- Goal: Find "good" feasible solution and tight bounds for z^*:

$$\bar{z} - \underline{z} \leq \varepsilon \text{ and } \varepsilon > 0 \text{ "small"}$$

Optimistic estimates of z^* from relaxations

- Either: Enlarge the set X by removing constraints
- Or: Replace $\mathbf{c}^{\mathrm{T}}\mathbf{x}$ by an underestimating function f, i.e., such that $f(\mathbf{x}) \leq \mathbf{c}^{\mathrm{T}}\mathbf{x}$ for all $\mathbf{x} \in X$
- Or: Do both

$\Rightarrow$ solve a *relaxation* of (ILP)

- Example (enlarge X):

  $X = \{\mathbf{x} \geq \mathbf{0} \mid \mathbf{A}\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \text{ integer } \}$ and
  $X^{\mathrm{LP}} = \{\mathbf{x} \geq \mathbf{0} \mid \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$

$$\Rightarrow \quad z^{\mathrm{LP}} = \min_{\mathbf{x} \in X^{\mathrm{LP}}} \mathbf{c}^{\mathrm{T}}\mathbf{x}$$

- It holds that $z^{\mathrm{LP}} \leq z^*$ since $X \subseteq X^{\mathrm{LP}}$

Relaxation principles that yield more tractable problems

- *Linear programming relaxation*
  Remove integrality requirements (enlarge X)

- *Combinatorial relaxation*
  E.g. remove subcycle constraints from asymmetric TSP $\Rightarrow$ min-cost assignment (enlarge X)

- *Lagrangean relaxation*
  Move "complicating" constraints to the objective function, with penalties for infeasible solutions; then find "optimal" penalties (enlarge X and construct a function f such that $f(\mathbf{x}) \leq \mathbf{c}^{\mathrm{T}}\mathbf{x}$, $\forall \mathbf{x} \in X$)

Tight bounds

- Suppose that $\bar{\mathbf{x}} \in X$ is a feasible solution to ILP (min-problem) and that $\underline{\mathbf{x}}$ solves a relaxation of ILP
- Then
$$\underline{z} := \mathbf{c}^{\mathrm{T}}\underline{\mathbf{x}} \leq z^* \leq \mathbf{c}^{\mathrm{T}}\bar{\mathbf{x}} =: \bar{z}$$
- $\underline{z}$ is an *optimistic* estimate of z^*
- $\bar{z}$ is a *pessimistic* estimate of z^*
- If $\bar{z} - \underline{z} \leq \varepsilon$ then the value of the solution candidate $\bar{\mathbf{x}}$ is at most ε from the optimal value z^*
- Efficient solution methods for ILP combine relaxation and heuristic methods to find tight bounds (small $\varepsilon \geq 0$)

Branch–&–Bound algorithms (B&B) (Ch. 15)

$$[\text{ILP}] \qquad z^* = \min_{\mathbf{x} \in X} \mathbf{c}^{\mathrm{T}} \mathbf{x}, \qquad X \subset Z^n$$

- Divide–and–conquer: a general principle to partition and search the feasible space
- Branch–&–Bound: Divide–and–conquer for finding *optimal* solutions to optimization problems with integrality requirements
- Can be adapted to different types of models
- Can be combined with other (e.g. heuristic) algorithms
- Also called implicit enumeration and tree search
- *Idea:* Enumerate all feasible solutions by a successive partitioning of X into a family of subsets
- Enumeration organized in a tree using graph search; it is made implicit by utilizing approximations of z^* from relaxations of [ILP] for cutting off branches of the tree

Branch–&–bound for ILP: Main concepts

- Relaxation: a simplification of [ILP] in which some constraints are removed
  - Purpose: to get simple (polynomially solvable) (node) subproblems, and optimistic approximations of z^*.
  - Examples: remove integrality requirements, remove or Lagrangean relax complicating (linear) constraints (e.g. sub-tour constraints)
- Branching strategy: rules for partitioning a subset of X
  - Purpose: exclude the solution to a relaxation if it is not feasible in [ILP]; corresponds to a *partitioning* of the feasible set
  - Examples: Branch on fractional values, subtours, etc.

B&B: Main concepts (continued)

- Tree search strategy: defines the order in which the nodes in the B&B tree are created and searched
  - Purpose: quickly find good feasible solutions; limit the size of the tree
  - Examples: depth-, breadth-, best-first.
- Node cutting criteria: rules for deciding when a subset should not be further partitioned
  - Purpose: avoid searching parts of the tree that cannot contain an optimal solution
  - Cut off a node if the corresponding node subproblem has
    - no feasible solution, or
    - an optimal solution which is feasible in [ILP], or
    - an optimal objective value that is worse (higher) than that of any known feasible solution

ILP: Solution by the branch–and–bound algorithm

- Relax integrality requirements ⇒ linear (continuous) program
- B&B tree: branch over fractional variable values