MVE165/MMG631
Linear and integer optimization with applications
Lecture 8
Combinatorial optimization theory and algorithms

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Overview

- Convexity
  - Local and global optima
- Heuristics:
  I Constructive heuristics
  II Local search methods
  III Approximation algorithms
  IV Meta-heuristics

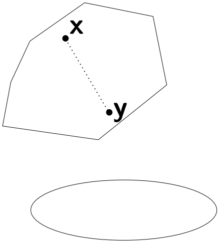
Convex sets

- A set S is convex if, for any elements $\mathbf{x}, \mathbf{y} \in S$ it holds that

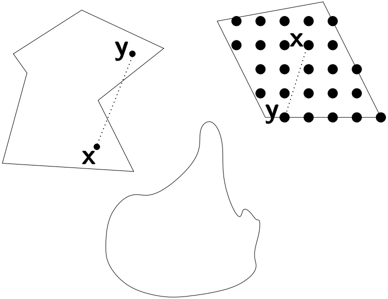
$$\alpha \mathbf{x} + (1 - \alpha)\mathbf{y} \in S \text{ for all } 0 \leq \alpha \leq 1$$

- Examples:

Convex sets



Non-convex sets



$\Rightarrow$ Integrality requirements $\Rightarrow$ nonconvex feasible set

Local vs. global optima

Consider a minimization problem:

$$\min_{\mathbf{x} \in X} \quad \mathbf{c}^{\mathrm{T}} \mathbf{x}$$

- Global optimum:
  A solution $\mathbf{x}^* \in X$ such that $\mathbf{c}^{\mathrm{T}} \mathbf{x}^* \leq \mathbf{c}^{\mathrm{T}} \mathbf{x}$ for all $\mathbf{x} \in X$
- ε-neighbourhood of $\bar{\mathbf{x}}$: $N_\varepsilon(\bar{\mathbf{x}}) = \left\{ \mathbf{x} \in X \,\middle|\, \|\mathbf{x} - \bar{\mathbf{x}}\| \leq \varepsilon \right\}$
- The distance measure $\|\mathbf{x} - \bar{\mathbf{x}}\|$ may be "freely" defined as, e.g., # arcs differing (Hamming distance), Euclidean, Manhattan, 2-interchange, ...
- Local optimum:
  A solution $\bar{\mathbf{x}} \in X$ such that $\mathbf{c}^{\mathrm{T}} \bar{\mathbf{x}} \leq \mathbf{c}^{\mathrm{T}} \mathbf{x}$ for all $\mathbf{x} \in N_\varepsilon(\bar{\mathbf{x}})$ for some $\varepsilon > 0$

Heuristic algorithms

- Optimization problems with high complexity may be too time consuming to solve to optimality
- Heuristic algorithms can be utilized
- But: Only local optimality can then be guaranteed

Heuristics I: Constructive heuristics (Ch. 16.3)

Consider a minimization problem:

$$\min_{\mathbf{x} \in X} \quad \mathbf{c}^{\mathrm{T}} \mathbf{x}$$

- Start by an "empty set" and "add" elements according to some (simple) rule
- Sometimes – no guarantee that even a feasible solution will be found
- No measure of how "close" to a global optimum a solution is
- Special rules for structured problems
- E.g. the greedy algorithm is a constructive heuristic (finds, however, optimal solution to minimum spanning tree)
- For TSP: nearest neighbour, cheapest insertion, farthest insertion, etc
- EXAMPLE!

Heuristics II: Local search (Ch. 16.4)

Consider a minimization problem:

$$\min_{\mathbf{x} \in X} \quad \mathbf{c}^{\mathrm{T}} \mathbf{x}$$

- Start from a feasible solution, which is iteratively improved by limited modifications
- Finds a local optimum
- No measure on how close to a global optimum a solution is
- Specialized for structured problems, but also general (Ch. 16.2)
- For TSP: e.g. 2-interchange, 3-interchange,
- EXAMPLE!

Local search heuristic algorithm (Ch. 16.4)

Consider a minimization problem:

$$\min_{\mathbf{x} \in X} \quad \mathbf{c}^{\mathrm{T}}\mathbf{x}$$

0. Initialization: Choose a feasible solution $\mathbf{x}^0 \in X$. Let $k = 0$.
1. Find all feasible points in an ε-neighbourhood $N_\varepsilon(\mathbf{x}^k)$ of $\mathbf{x}^k$
2. If $\mathbf{c}^{\mathrm{T}}\mathbf{x} \geq \mathbf{c}^{\mathrm{T}}\mathbf{x}^k$ for all $\mathbf{x} \in X \cap N_\varepsilon(\mathbf{x}^k) \Rightarrow$ Stop. $\mathbf{x}^k$ is a local optimum (w.r.t. N_ε)
3. Choose $\mathbf{x}^{k+1} \in X \cap N_\varepsilon(\mathbf{x}^k)$ such that $\mathbf{c}^{\mathrm{T}}\mathbf{x}^{k+1} < \mathbf{c}^{\mathrm{T}}\mathbf{x}^k$
4. Let $k := k + 1$ and go to step 1

Heuristics III: Approximation algorithms (Ch. 16.6)

Consider a minimization problem:

$$\min_{\mathbf{x} \in X} \quad \mathbf{c}^{\mathrm{T}}\mathbf{x}$$

- Performance guarantee: $\dfrac{\bar{z} - z^*}{z^*} \leq \alpha$ for some $0 < \alpha \leq 1$
- Specialized algorithms for structured problems

Example of an approximation algorithm

- The spanning tree approximation algorithm for the TSP
- Need some more definitions for this: Spanning trees and greedy algorithms

The minimum spanning tree (MST) problem

- Given an undirected graph $G = (N, E)$ with nodes N, edges E and distances d_{ij} for each edge $(i, j) \in E$
- Find a subset of the edges that connects all nodes at minimum total distance
- The number of edges in a spanning tree is $|N| - 1$
- A (spanning) tree contains *no cycles*
- MST is a very simple problem (a matroid) that can be solved by *greedy algorithms*

Greedy algorithms for MST

- Prim's algorithm

  1. Start at an arbitrary node
  2. Among the nodes that are not yet connected, choose the one that can be connected at minimum cost
  3. Stop when all nodes are connected

- SOLVE AN EXAMPLE!

- Kruskal's algorithm

  1. Sort the edges by increasing distances
  2. Choose edges starting from the beginning of the list; skip edges resulting in cycles
  3. Stop when all nodes are connected

- SOLVE AN EXAMPLE!

Spanning tree approximation algorithm for the TSP

- Consider a TSP on an undirected graph $G = (N, E, \mathbf{c})$
- Assume
  - G complete $\Leftrightarrow$ edges between all pairs of nodes
  - Δ-inequality: $c_{ij} \leq c_{ik} + c_{kj}$ for all $i, j, k \in N$ DRAW!

1. Find a minimum spanning tree $T \subset E$ on G
2. Create a multigraph G' using *two copies* of each edge in T
3. Find an Eulerian walk of G' and and embedded TSP-tour

- Guarantee: $\dfrac{\bar{z} - z^*}{z^*} \leq 1$
- Not worse than twice the optimal tour!
- EXAMPLE!

Proof

- Let $c(\text{TSP}) = z^*$ and $c(\text{tour}) = \bar{z}$
- A spanning tree is a relaxation of a TSP:
  All soubtour elimination constraints are fulfilled, but not the node valence (2 edges incident to each node)

$\Rightarrow$ $c(\text{MST}) \leq c(\text{TSP})$

- Two copies of each edge $\Rightarrow c(\text{tour}) \leq 2c(\text{MST}) \leq 2c(\text{TSP})$

$$\Rightarrow \frac{c(\text{tour}) - c(\text{TSP})}{c(\text{TSP})} \leq 1$$

Heuristics IV: Metaheuristics (Ch. 16.5)

Consider a minimization problem:

$$\min_{\mathbf{x} \in X} \quad \mathbf{c}^{\mathrm{T}} \mathbf{x}$$

- Intends to be more efficient than just plain local search methods
- Includes tabu search, simulated annealing

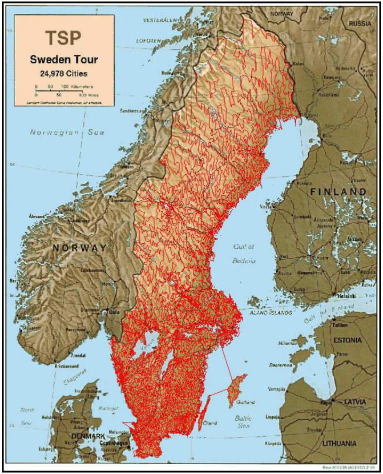
More about heuristics

- Start using a constructive heuristic $\Rightarrow$ feasible solution
- The choice of definition of a neighbourhood is model specific (e.g. Euclidean distance, number of arcs differing,)
- Apply a local search algorithm
- Finds a *locally* optimal solution
- *No guarantee* to find global optimal solutions
- Extensions (e.g. tabu search): Temporarily allow worse solutions to “move away” from a local optimum (Ch. 16.5)
- Larger neighbourhoods yield better local optima, but takes more computation time to explore

The historical development of TSP solution

- Optimal solutions to TSP's of different sizes found

year	$n =$
1954	49
1962	33
1977	120
1987	532
1987	666
1987	2392
1994	7397
1998	13509
2001	15112
2004	24978
2005/06	85900



The worlds largest TSP solved "so far" (2004) ...

- A TSP of 24 978 cities and villages (red houses) in Sweden
- Optimal tour: $\approx$ 72 500 km (855597 TSP LIB units)
- The tour of length 855 597 was found in March 2003 (Lin-Kernighan's TSP heuristic)
- It was proven in May 2004 that no shorter tour exists
- A variety of heuristics, B&B, and cut generation algorithms
- The final stages that improved the lower bound from 855 595 up to 855 597 required $\approx$ 8 years of computation time (running in parallel on a network of Linux workstations)

  "Without knowledge of the 855 597 tour we would not have made the decision to carry out this final computation"
- `www.tsp.gatech.edu`
- New record in 2005/06: 85 900 locations in a VLSI application `www.tsp.gatech.edu/pla85900`
- Curious: iPhone/iPad App: Concorde TSP