

MVE165/MMG631
Linear and integer optimization with applications
Lecture 9a and 10b
Multiobjective optimization

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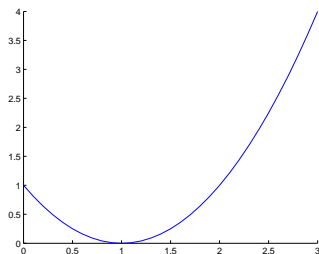
Applied optimization — multiple objectives

- ▶ Many practical optimization problems have several objectives which may be in conflict
- ▶ Some goals cannot be reduced to a common scale of cost/profit \Rightarrow trade-offs must be addressed
- ▶ **Examples**
 - ▶ Financial investments — risk vs. return
 - ▶ Engine design — efficiency vs. NO_x vs. soot
 - ▶ Wind power production — investment vs. operation (Ass 3a)
- ▶ **Literature on multiple objectives' optimization**

Copies from the book *Optimization in Operations Research* by R.L. Rardin (1998) pp. 373–387, handed out (on paper, copies kept outside Ann-Brith's office, room MVL2087)

Optimization of multiple objectives

- ▶ Consider the minimization of $f(x) = (x - 1)^2$ subject to $0 \leq x \leq 3$
- ▶ Optimal solution: $x^* = 1$



Optimization of multiple objectives

- ▶ Consider then two objectives:

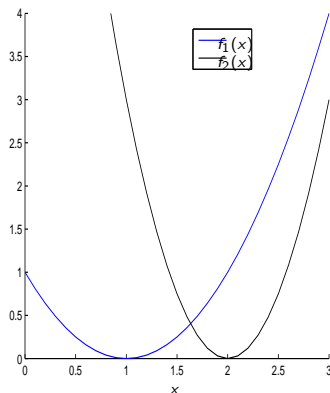
$$\text{minimize } [f_1(x), f_2(x)]$$

$$\text{subject to } 0 \leq x \leq 3$$

where

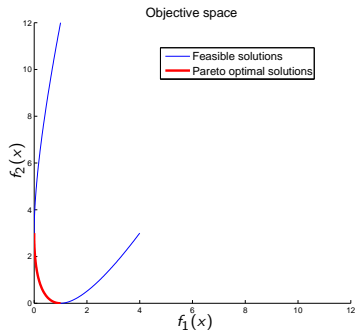
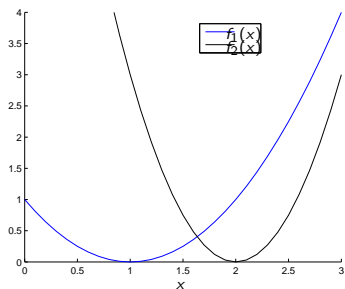
$$f_1(x) = (x - 1)^2, \quad f_2(x) = 3(x - 2)^2$$

- ▶ How can an optimal solution be defined?
 - ▶ A solution is **Pareto optimal** if **no other** feasible solution has a better value in **all** objectives
- ⇒ All points $x \in [1, 2]$ are Pareto optimal



Pareto optimal solutions in the objective space

- ▶ minimize $[f_1(x), f_2(x)]$ subject to $0 \leq x \leq 3$
where $f_1(x) = (x - 1)^2$ and $f_2(x) = 3(x - 2)^2$
- ▶ A solution is **Pareto optimal** if **no other** feasible solution has a better value in **all** objectives

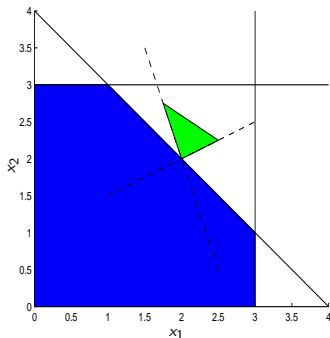


- ▶ Pareto optima \Leftrightarrow nondominated points \Leftrightarrow efficient frontier

Efficient points

- ▶ Consider a bi-objective linear program:

$$\begin{array}{ll} \text{maximize} & 3x_1 + x_2 \\ \text{maximize} & -x_1 + 2x_2 \\ \text{subject to} & x_1 + x_2 \leq 4 \\ & 0 \leq x_1 \leq 3 \\ & 0 \leq x_2 \leq 3 \end{array}$$

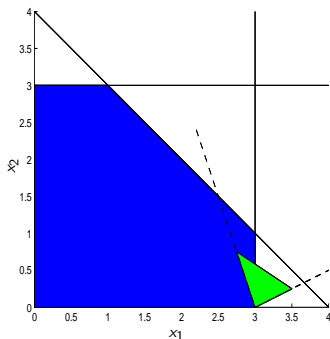


- ▶ The solutions in the green cone are better than the solution $(2, 2)$ w.r.t. both objectives
- ▶ The point $x = (2, 2)$ is an *efficient*, or *non-dominated*, solution

Dominated points



$$\begin{array}{ll} \text{maximize} & 3x_1 + x_2 \\ \text{maximize} & -x_1 + 2x_2 \\ \text{subject to} & x_1 + x_2 \leq 4 \\ & 0 \leq x_1 \leq 3 \\ & 0 \leq x_2 \leq 3 \end{array}$$

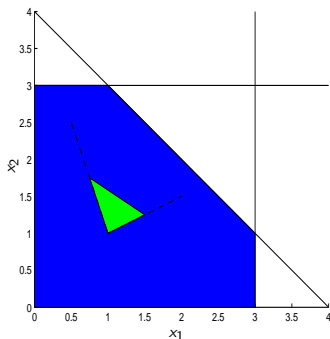


- ▶ The point $x = (3, 0)$ is *dominated* by the solutions in the green cone
- ▶ Feasible solutions exist that are better w.r.t. both objectives

Dominated points



$$\begin{aligned} &\text{maximize} && 3x_1 + x_2 \\ &\text{maximize} && -x_1 + 2x_2 \\ &\text{subject to} && x_1 + x_2 \leq 4 \\ &&& 0 \leq x_1 \leq 3 \\ &&& 0 \leq x_2 \leq 3 \end{aligned}$$

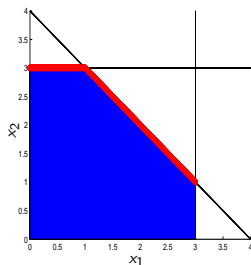


- ▶ The point $x = (1, 1)$ is dominated by the solutions in the green cone
- ▶ Feasible solutions exist that are better w.r.t. both objectives

The efficient frontier—the set of Pareto optimal solutions



$$\begin{aligned} &\text{maximize} && 3x_1 + x_2 \\ &\text{maximize} && -x_1 + 2x_2 \\ &\text{subject to} && x_1 + x_2 \leq 4 \\ &&& 0 \leq x_1 \leq 3 \\ &&& 0 \leq x_2 \leq 3 \end{aligned}$$



- ▶ The set of efficient solutions is given by

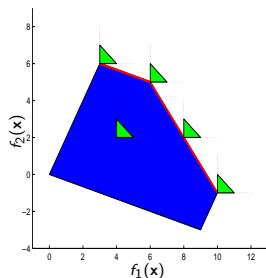
$$\left\{ \mathbf{x} \in \mathbb{R}^2 \mid \mathbf{x} = \alpha \begin{pmatrix} 3 \\ 1 \end{pmatrix} + (1 - \alpha) \begin{pmatrix} 1 \\ 3 \end{pmatrix}, 0 \leq \alpha \leq 1 \right\} \cup \left\{ \mathbf{x} \in \mathbb{R}^2 \mid \mathbf{x} = \alpha \begin{pmatrix} 1 \\ 3 \end{pmatrix} + (1 - \alpha) \begin{pmatrix} 0 \\ 3 \end{pmatrix}, 0 \leq \alpha \leq 1 \right\}$$

Note that this is *not* a convex set!

The Pareto optimal set in the objective space



$$\begin{aligned} &\text{maximize} && f_1(\mathbf{x}) := 3x_1 + x_2 \\ &\text{maximize} && f_2(\mathbf{x}) := -x_1 + 2x_2 \\ &\text{subject to} && x_1 + x_2 \leq 4 \\ &&& 0 \leq x_1 \leq 3 \\ &&& 0 \leq x_2 \leq 3 \end{aligned}$$

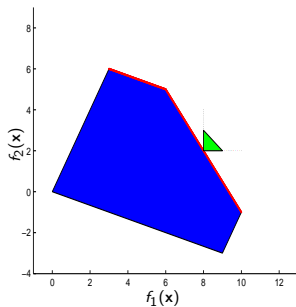
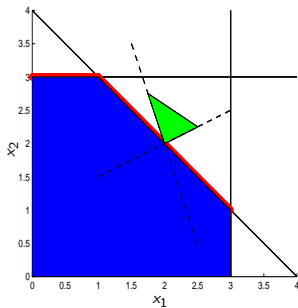


- ▶ The set of Pareto optimal objective values is given by

$$\left\{ (f_1, f_2) \in \mathbb{R}^2 \mid \mathbf{f} = \alpha \begin{pmatrix} 10 \\ -1 \end{pmatrix} + (1 - \alpha) \begin{pmatrix} 6 \\ 5 \end{pmatrix}, 0 \leq \alpha \leq 1 \right\} \cup \left\{ (f_1, f_2) \in \mathbb{R}^2 \mid \mathbf{f} = \alpha \begin{pmatrix} 6 \\ 5 \end{pmatrix} + (1 - \alpha) \begin{pmatrix} 3 \\ 6 \end{pmatrix}, 0 \leq \alpha \leq 1 \right\}$$

Mapping from the decision space to the objective space

maximize $[3x_1 + x_2; -x_1 + 2x_2]$
subject to $x_1 + x_2 \leq 4, \quad 0 \leq x_1 \leq 3, \quad 0 \leq x_2 \leq 3$



Solutions methods for multiobjective optimization

- ▶ Construct the efficient frontier by treating one objective as a constraint and optimizing for the other:

$$\begin{array}{ll} \text{maximize} & 3x_1 + x_2 \\ \text{subject to} & -x_1 + 2x_2 \geq \varepsilon \\ & x_1 + x_2 \leq 4 \\ & 0 \leq x_1 \leq 3 \\ & 0 \leq x_2 \leq 3 \end{array}$$

- ▶ Here, let $\varepsilon \in [-1, 6]$. Why?
- ▶ What if the number of objectives is ≥ 3 ?
- ▶ How many single objective linear programs do we have to solve for seven objectives and ten values of ε_k for each objective f_k , $k = 1, \dots, 7$?

Solution methods: preemptive optimization

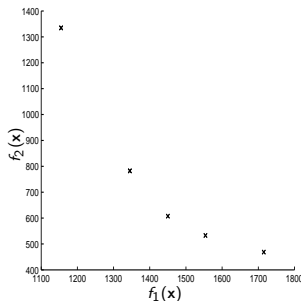
- ▶ Consider one objective at a time—the most important first
- ▶ Solve for the first objective
- ▶ Solve for the second objective over the solution set for the first
- ▶ Solve for the third objective over the solution set for the second
- ▶ ...

- ▶ The solution is an efficient point
- ▶ But: Different orderings of the objectives yield different solutions

- ▶ Exercise: solve the previous example using preemptive optimization for different orderings of the objective functions

Solution methods: weighted sums of objectives

- ▶ Give each maximization (minimization) objective a positive (negative) weight
- ▶ Solve a single objective maximization problem
- ⇒ Yields an efficient solution
- ▶ Well spread weights do not necessarily produce solutions that are well spread on the efficient frontier (ex: $\{\frac{1}{10}, \frac{1}{2}, 1, 2, 10\}$)
- ▶ If the objectives are *not* concave (maximization) or the feasible set is *not* convex, as, e.g., integrality constrained, then not all points on the efficient frontier may be possible to detect using weighted sums of objectives



Solution methods: soft constraints

- ▶ Consider the multiobjective optimization problem to

$$\text{maximize } [f_1(\mathbf{x}), \dots, f_K(\mathbf{x})] \text{ subject to } \mathbf{x} \in X$$

- ▶ Define a target value t_k and a deficiency variable $d_k \geq 0$ for each objective f_k
- ▶ Construct a soft constraint for each objective:

$$\text{maximize } f_k(\mathbf{x}) \quad \Rightarrow \quad f_k(\mathbf{x}) + d_k \geq t_k, \quad k = 1, \dots, K$$

- ▶ Minimize the sum of deficiencies:

$$\begin{aligned} &\text{minimize} && \sum_{k \in K} d_k \\ &\text{subject to} && f_k(\mathbf{x}) + d_k \geq t_k, \quad k = 1, \dots, K \\ &&& d_k \geq 0, \quad k = 1, \dots, K \\ &&& \mathbf{x} \in X \end{aligned}$$

- ▶ Important: Find first a common scale for f_k , $k = 1, \dots, K$

Normalizing the objectives

- ▶ Consider the multiobjective optimization problem to

maximize $[f_1(\mathbf{x}), \dots, f_K(\mathbf{x})]$ subject to $\mathbf{x} \in X$

- ▶ Let

$$\tilde{f}_k(\mathbf{x}) = \frac{f_k(\mathbf{x})}{f_k^{\max} - f_k^{\min}}, \quad k = 1, \dots, K,$$

where $f_k^{\max} = \max_{\mathbf{x} \in X} f_k(\mathbf{x})$ and $f_k^{\min} = \min_{\mathbf{x} \in X} f_k(\mathbf{x})$.

- ▶ Then, $\tilde{f}_k(\mathbf{x}) \in [0, 1]$ for all $\mathbf{x} \in X$, so that the functions \tilde{f}_k can be compared in a common scale.