MVE165/MMG631
Linear and integer optimization with applications
Lecture 9a and 10b
Multiobjective optimization

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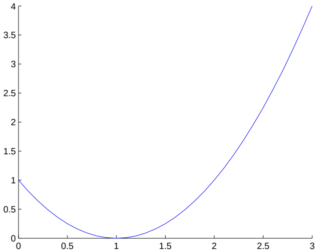
Applied optimization — multiple objectives

- Many practical optimization problems have several objectives which may be in conflict
- Some goals cannot be reduced to a common scale of cost/profit $\Rightarrow$ trade-offs must be addressed
- **Examples**
  - Financial investments — risk vs. return
  - Engine design — efficiency vs. NO_x vs. soot
  - Wind power production — investment vs. operation (Ass 3a)
- **Literature on multiple objectives' optimization**
  Copies from the book *Optimization in Operations Research* by R.L. Rardin (1998) pp. 373–387, handed out (on paper, copies kept outside Ann-Brith's office, room MVL2087)

Optimization of multiple objectives

- Consider the minimization of $f(x) = (x-1)^2$ subject to $0 \leq x \leq 3$

- Optimal solution: $x^* = 1$



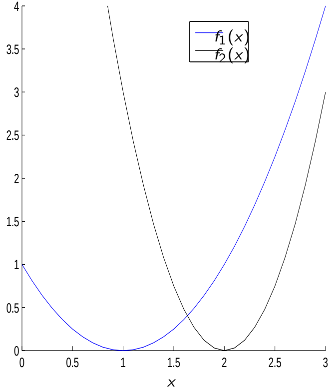
Optimization of multiple objectives

- Consider then two objectives:

$$\text{minimize } [f_1(x), f_2(x)]$$
$$\text{subject to } 0 \leq x \leq 3$$

where

$$f_1(x) = (x-1)^2, \ \ f_2(x) = 3(x-2)^2$$

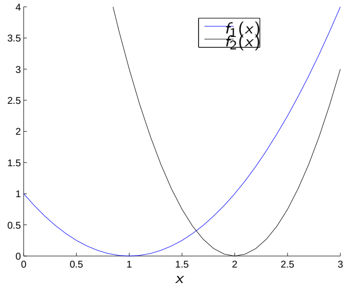


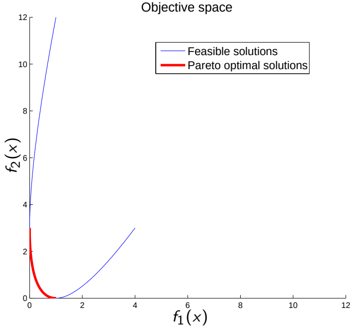
- How can an optimal solution by defined?
- A solution is **Pareto optimal** if **no other** feasible solution has a better value in **all** objectives

$\Rightarrow$ All points $x \in [1, 2]$ are Pareto optimal

Pareto optimal solutions in the objective space

- minimize $[f_1(x), f_2(x)]$ subject to $0 \leq x \leq 3$
  where $f_1(x) = (x-1)^2$ and $f_2(x) = 3(x-2)^2$
- A solution is **Pareto optimal** if **no other** feasible solution has a better value in **all** objectives



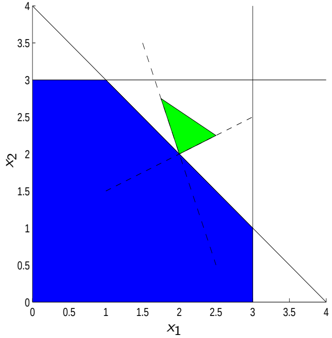


- Pareto optima $\Leftrightarrow$ nondominated points $\Leftrightarrow$ efficient frontier

Efficient points

- Consider a bi-objective linear program:

$$\begin{aligned} \text{maximize} \quad & 3x_1 + x_2 \\ \text{maximize} \quad & -x_1 + 2x_2 \\ \text{subject to} \quad & x_1 + x_2 \leq 4 \\ & 0 \leq x_1 \leq 3 \\ & 0 \leq x_2 \leq 3 \end{aligned}$$



- The solutions in the green cone are better than the solution $(2,2)$ w.r.t. both objectives
- The point $x = (2,2)$ is an *efficient, or non-dominated,* solution

Dominated points

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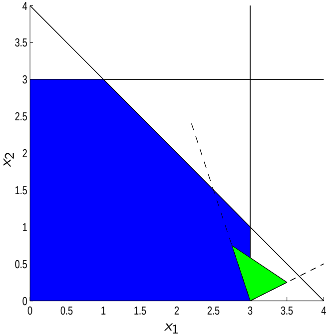
$$\begin{array}{rl} \text{maximize} & 3x_1 + x_2 \\ \text{maximize} & -x_1 + 2x_2 \\ \text{subject to} & x_1 + x_2 \leq 4 \\ & 0 \leq x_1 \leq 3 \\ & 0 \leq x_2 \leq 3 \end{array}$$

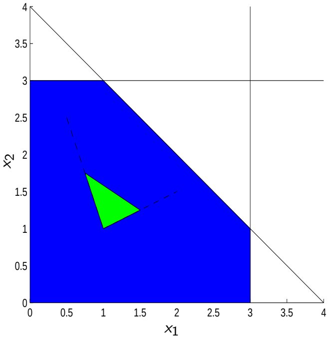


- The point $x = (3, 0)$ is *dominated* by the solutions in the green cone
- Feasible solutions exist that are better w.r.t. both objectives

Dominated points

-

$$
\begin{array}{rl}
\text{maximize} & 3x_1 + x_2 \\
\text{maximize} & -x_1 + 2x_2 \\
\text{subject to} & x_1 + x_2 \leq 4 \\
& 0 \leq x_1 \leq 3 \\
& 0 \leq x_2 \leq 3
\end{array}
$$



- The point $x = (1, 1)$ is dominated by the solutions in the green cone
- Feasible solutions exist that are better w.r.t. both objectives

The efficient frontier—the set of Pareto optimal solutions

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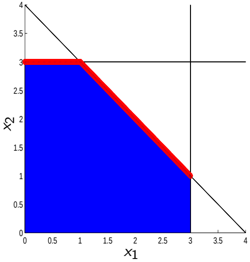
$$
\begin{array}{ll}
\text{maximize} & 3x_1 + x_2 \\
\text{maximize} & -x_1 + 2x_2 \\
\text{subject to} & x_1 + x_2 \leq 4 \\
 & 0 \leq x_1 \leq 3 \\
 & 0 \leq x_2 \leq 3
\end{array}
$$



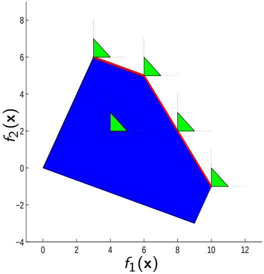
- The set of efficient solutions is given by

$$
\left\{ \mathbf{x} \in \Re^2 \,\middle|\, \mathbf{x} = \alpha \begin{pmatrix} 3 \\ 1 \end{pmatrix} + (1-\alpha) \begin{pmatrix} 1 \\ 3 \end{pmatrix}, 0 \leq \alpha \leq 1 \right\} \bigcup
\left\{ \mathbf{x} \in \Re^2 \,\middle|\, \mathbf{x} = \alpha \begin{pmatrix} 1 \\ 3 \end{pmatrix} + (1-\alpha) \begin{pmatrix} 0 \\ 3 \end{pmatrix}, 0 \leq \alpha \leq 1 \right\}
$$

Note that this is *not* a convex set!

The Pareto optimal set in the objective space

- $$\begin{aligned} \text{maximize} \quad & f_1(\mathbf{x}) := 3x_1 + x_2 \\ \text{maximize} \quad & f_2(\mathbf{x}) := -x_1 + 2x_2 \\ \text{subject to} \quad & x_1 + x_2 \leq 4 \\ & 0 \leq x_1 \leq 3 \\ & 0 \leq x_2 \leq 3 \end{aligned}$$

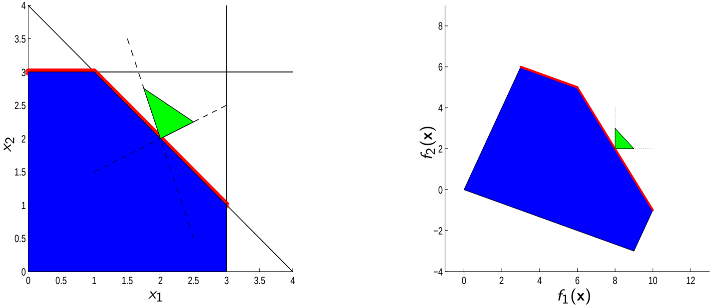


- The set of Pareto optimal objective values is given by

$$\left\{ (f_1, f_2) \in \Re^2 \;\middle|\; \mathbf{f} = \alpha \begin{pmatrix} 10 \\ -1 \end{pmatrix} + (1-\alpha) \begin{pmatrix} 6 \\ 5 \end{pmatrix}, 0 \leq \alpha \leq 1 \right\} \bigcup$$
$$\left\{ (f_1, f_2) \in \Re^2 \;\middle|\; \mathbf{f} = \alpha \begin{pmatrix} 6 \\ 5 \end{pmatrix} + (1-\alpha) \begin{pmatrix} 3 \\ 6 \end{pmatrix}, 0 \leq \alpha \leq 1 \right\}$$

Mapping from the decision space to the objective space

$$\begin{array}{ll} \text{maximize} & [3x_1 + x_2; -x_1 + 2x_2] \\ \text{subject to} & x_1 + x_2 \leq 4, \quad 0 \leq x_1 \leq 3, \quad 0 \leq x_2 \leq 3 \end{array}$$



Solutions methods for multiobjective optimization

- Construct the efficient frontier by treating one objective as a constraint and optimizing for the other:

$$\begin{aligned} \text{maximize} \quad & 3x_1 + x_2 \\ \text{subject to} \quad & -x_1 + 2x_2 \geq \varepsilon \\ & x_1 + x_2 \leq 4 \\ & 0 \leq x_1 \leq 3 \\ & 0 \leq x_2 \leq 3 \end{aligned}$$

- Here, let $\varepsilon \in [-1, 6]$. Why?
- What if the number of objectives is ≥ 3?
- How many single objective linear programs do we have to solve for seven objectives and ten values of ε_k for each objective f_k, $k = 1, \ldots, 7$?

Solution methods: preemptive optimization

- Consider one objective at a time—the most important first
- Solve for the first objective
- Solve for the second objective over the solution set for the first
- Solve for the third objective over the solution set for the second
- ...

- The solution is an efficient point
- But: Different orderings of the objectives yield different solutions

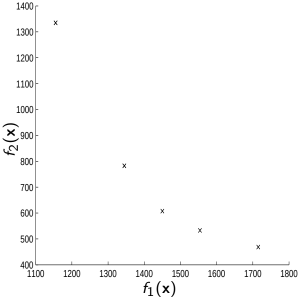
- Exercise: solve the previous example using preemptive optimization for different orderings of the objective functions

Solution methods: weighted sums of objectives

- Give each maximization (minimization) objective a positive (negative) weight
- Solve a single objective maximization problem

$\Rightarrow$ Yields an efficient solution

- Well spread weights do not necessarily produce solutions that are well spread on the efficient frontier (ex: $\{\frac{1}{10}, \frac{1}{2}, 1, 2, 10\}$)
- If the objectives are *not* concave (maximization) or the feasible set is *not* convex, as, e.g., integrality constrained, then not all points on the efficient frontier may be possible to detect using weighted sums of objectives
- [ILLUSTRATING EXAMPLE ON THE BOARD]



Solution methods: soft constraints

- Consider the multiobjective optimization problem to

$$\text{maximize } [f_1(\mathbf{x}), \dots, f_K(\mathbf{x})] \text{ subject to } \mathbf{x} \in X$$

- Define a target value t_k and a deficiency variable $d_k \geq 0$ for each objective f_k
- Construct a soft constraint for each objective:

$$\text{maximize } f_k(\mathbf{x}) \quad \Rightarrow \quad f_k(\mathbf{x}) + d_k \geq t_k, \quad k = 1, \dots, K$$

- Minimize the sum of deficiencies:

$$\begin{aligned} \text{minimize} \quad & \sum_{k \in K} d_k \\ \text{subject to} \quad & f_k(\mathbf{x}) + d_k \geq t_k, \quad k = 1, \dots, K \\ & d_k \geq 0, \quad k = 1, \dots, K \\ & \mathbf{x} \in X \end{aligned}$$

- Important: Find first a common scale for f_k, $k = 1, \dots, K$

Normalizing the objectives

- Consider the multiobjective optimization problem to

$$\text{maximize } [f_1(\mathbf{x}), \ldots, f_K(\mathbf{x})] \text{ subject to } \mathbf{x} \in X$$

- Let

$$\tilde{f}_k(\mathbf{x}) = \frac{f_k(\mathbf{x}) - f_k^{\min}}{f_k^{\max} - f_k^{\min}}, \quad k = 1, \ldots, K,$$

where $f_k^{\max} = \max_{\mathbf{x} \in X} \{f_k(\mathbf{x})\}$ and $f_k^{\min} = \min_{\mathbf{x} \in X} \{f_k(\mathbf{x})\}$.

- Then, $\tilde{f}_k(\mathbf{x}) \in [0, 1]$ for all $\mathbf{x} \in X$, so that the functions $\tilde{f}_k$ can be compared in a common scale.