# Assignment 3a: The Traveling Salesman Problem

## MVE165/MMG631: Linear and integer optimisation with applications

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# Traveling Salesman Problem (TSP)

- One of the most studied problems in the area of optimization.
- The name is a mystery, but gives a clear connection to the applications of the problem.
- 1832, handbook *Der Handlungsreisende* for traveling salesmen.
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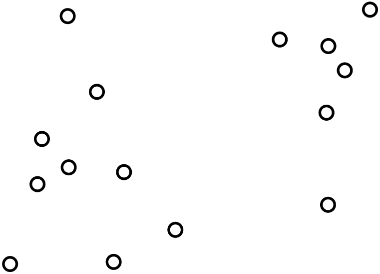
**Definition: Traveling Salesman Problem.**

*Given a list of cities and their pairwise distances, find the shortest possible tour that visits each city exactly once.*

# Example



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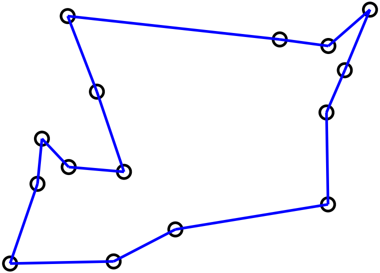

# Applications

The TSP has many applications.

- Logistics.
- Production (microchips).
- DNA-sequencing.
- Agriculture.
- Internet planning.

# Complexity

The TSP is a *NP-complete* problem (the decision version of it).

- No polynomial algorithm for solving it to optimality.
- Exponential in the number of cities.
- $(N-1)!$ different tours.

Large problems solved to optimality

- VLSI problem (85,900 nodes). Solved 2004, first studied 1991.
- Shortest tour between all the cities in Sweden (24,978 cities) found in 2001. Length $\approx$ 72,500 km

# Special cases

Let $c_{ij}$ be the distance from city $i$ to city $j$.

**Symmetric TSP (undirected graph)**

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## Symmetric TSP (undirected graph)

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## Metric TSP (triangle inequality satisfied)

$$c_{ik} + c_{kj} \geq c_{ij}, \quad \forall \text{ cities } i, j, k$$

# Integer linear program (symmetric TSP)

- Consider a set $\mathcal{N} = \{1, \ldots, N\}$ of cities.
- Let $c_{ij}$ be the distance from city $i$ to city $j$.
- Let $\mathcal{L} \subseteq \mathcal{N} \times \mathcal{N}$ denote the undirected links in the graph. We use $\mathcal{L} = \{(i,j) : i,j \in \mathcal{N}, i < j\}$
- Introduce binary variables $x_{ij}$ where

$$x_{ij} = \begin{cases} 1 & \text{if there is a connection between city } i \text{ and city } j \\ 0 & \text{otherwise} \end{cases}$$

# Linear integer program (symmetric TSP)

$$
\begin{aligned}
\text{minimize} \quad & \sum_{(i,j)\in\mathcal{L}} c_{ij}x_{ij}, \\
\text{subject to} \quad & \sum_{j\in\mathcal{N}:(i,j)\in\mathcal{L}} x_{ij} + \sum_{j\in\mathcal{N}:(j,i)\in\mathcal{L}} x_{ji} = 2 \quad , \qquad i \in \mathcal{N} \\
& \sum_{(i,j)\in\mathcal{L}:\{i,j\}\in\mathcal{S}} x_{ij} \le |\mathcal{S}| - 1, \ \forall \mathcal{S} \subset \mathcal{N} : 2 \le |\mathcal{S}| \le N - 2 \\
& x_{ij} \in \{0, 1\} \quad , \qquad i, j \in \mathcal{N}.
\end{aligned}
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- Objective is to minimize the total length of the tour.
- First constraints makes sure that we visit each city once.
- Second constraint makes sure that no subtours are allowed.

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- From each subset of cities, we must travel at least once to another city not included in the subset.

$$\sum_{(i,j) \in \mathcal{L} : i \in \mathcal{S}, j \in \mathcal{N} \setminus \mathcal{S}} x_{ij} + \sum_{(j,i) \in \mathcal{L} : i \in \mathcal{S}, j \in \mathcal{N} \setminus \mathcal{S}} x_{ji} \geq 2, \quad \forall \mathcal{S} \subset \mathcal{N} : 2 \leq |\mathcal{S}| \leq N - 2.$$

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One problem with the model is that the number of subtour constraints in both formulations are $\approx 2^N$.

# Algorithms

Exact algorithms.

- *Brute force technique.* Enumerate all possible tours, choose the shortest one. $\sim \mathcal{O}(N!)$
- *Held-Karp algorithm.* Use dynamic programming. $\sim \mathcal{O}(N^2 2^N)$

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Have to use

- *Heuristics*: Find feasible and acceptable solutions.
- *Relaxation algorithms*: Produce lower bounds on optimal objective value.

Together these algorithms can provide us with an *optimality interval*

# Construcive heuristics

Strategies, rules for obtaining feasible solutions.

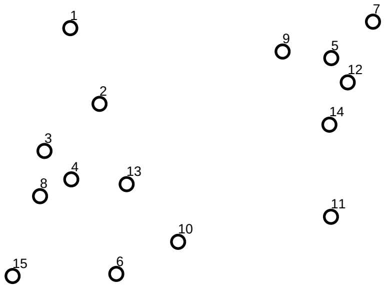
*Deterministic*

- Based on simple rules for choosing tours: *Nearest neighbour*, *Insertion heuristics*
- Based on solving easier subproblems. *MST-heuristic*, *Christophides heuristic*

*Probabilistic*

- Based on stochastic rules for choosing tours.
- *Genetic algorithms*, *simulated annealing*, *ant colony optimization*.

# Nearest neighbour heuristic



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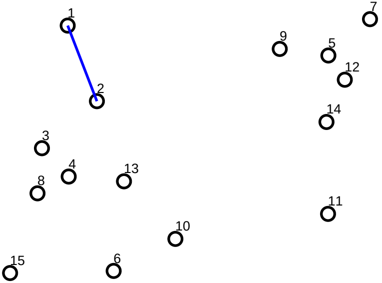

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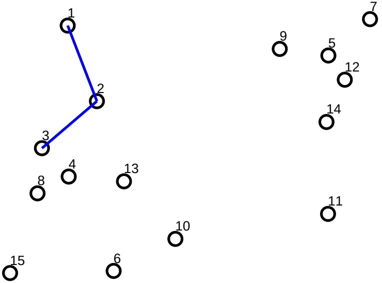

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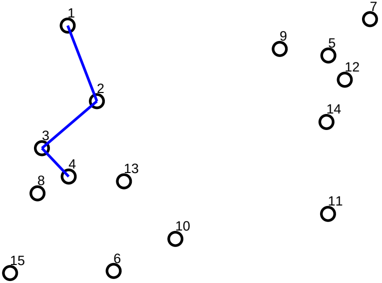

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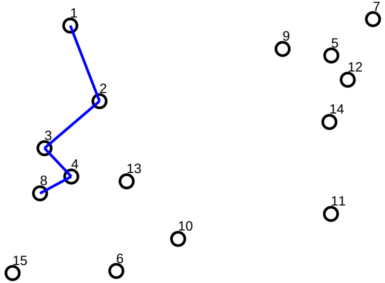

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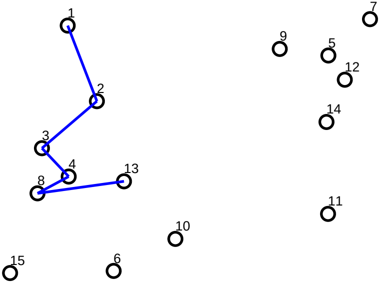

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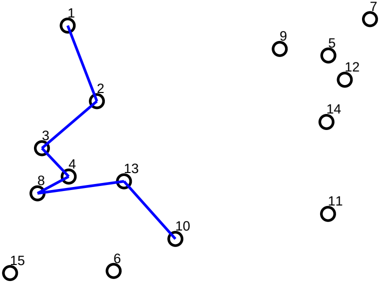

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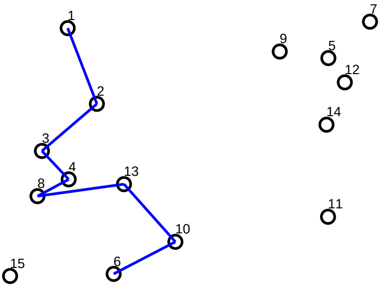

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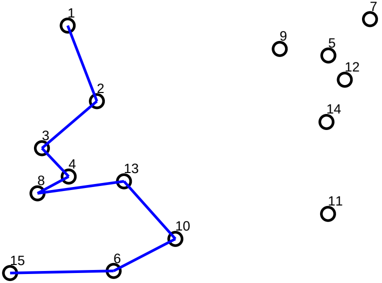

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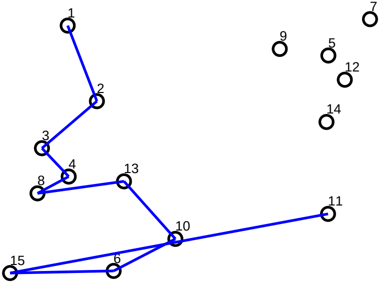

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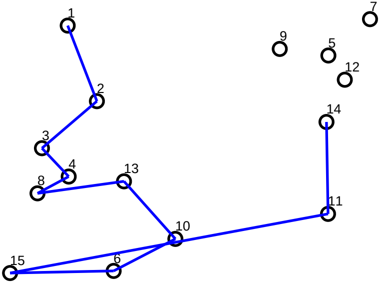

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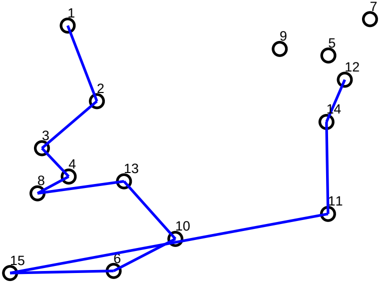

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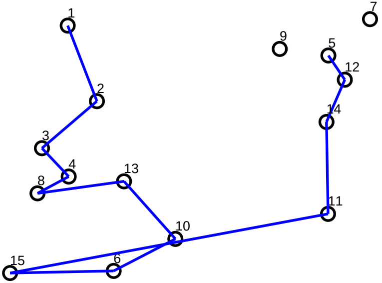

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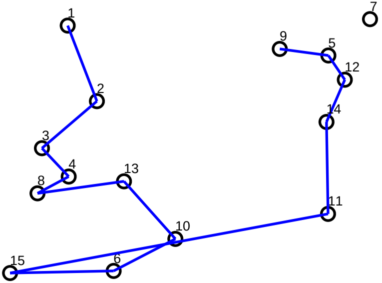

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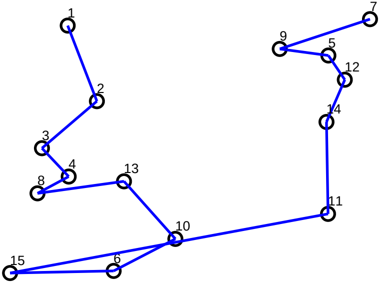

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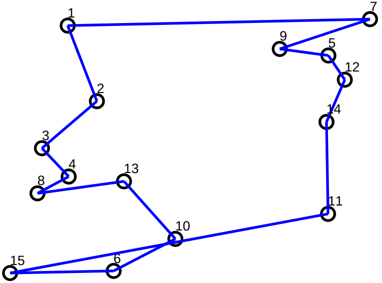

# Improvement heuristics

Algorithms for improving a feasible solution. Local search heuristics. Utilize the fact that we are considering *metric* TSP problems.

Examples: *k-opt heuristics*, *crossing elimination*

# Relaxation algorithms

Gives lower bounds on objective value. Trick is to relax the problem such that

- the reduced problem is easy to solve, and
- the lower bound given by the relaxation is good.

Tradeoff between the two objectives.

Examples: *Branch and bound*, *Cutting plane methods*, *1-tree Lagrangian relaxation*.

# 1-tree Lagrangian relaxation

Idea:

- Lagrangian relax the assignment constraints

$$\sum_{j \in \mathcal{N}:(i,j) \in \mathcal{L}} x_{ij} + \sum_{j \in \mathcal{N}:(j,i) \in \mathcal{L}} x_{ji} = 2 \quad i \in \mathcal{N}$$

  for all nodes except one, say node $s$. This means assigning a Lagrangian multiplier to each node (node price).

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- Iteratively updating the node prices such that we increase our lower bound in each iteration.

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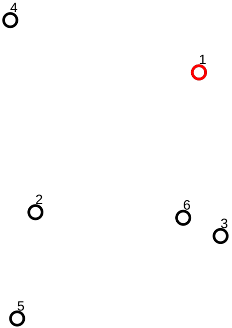

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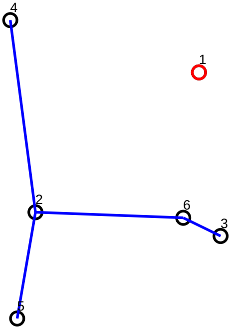

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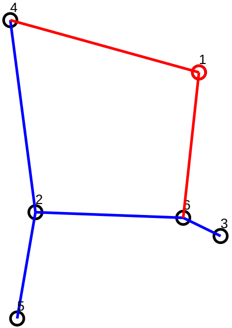

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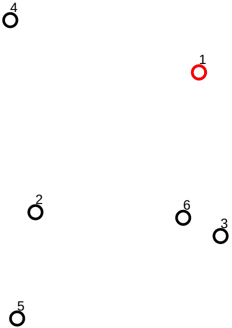

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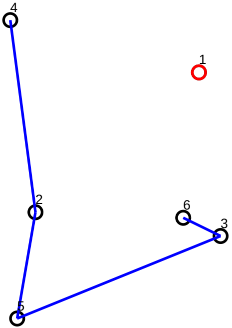

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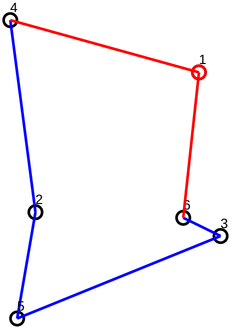

# Assignment 3a: The Traveling Salesman Problem

What do you do in the assignment?

- You get familiar with one of the most studied problems in optimization.
- You will use CPLEX to solve some small problems.
- You develop and implement different algorithms, both heuristics and relaxation algorithms.
- You get more familiar with *either* the theory of relaxation algorithms *or* probabilistic heuristics.

# Optimal tour of Sweden