MVE165/MMG631
Linear and Integer Optimization with Applications
Lecture 1
Introduction; course map; operations research; modelling; graphic solution

Ann-Brith Strömberg

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Staff

- **Examiner and main lecturer**
  - Ann-Brith Strömberg (anstr@chalmers.se, room L2087)
- **Course assistant**
  - Zuzana Šabartová (zuzana@chalmers.se, room L2099)
- **Problem solving sessions**
  - Zuzana Šabartová (zuzana@chalmers.se, room L2099)
  - Kin Cheong Sou (kincheong@chalmers.se, room L2100)
- **Guest lecturers**
  - Emil Gustavsson (Mathematical Sciences)
  - Ola Carlson (Energy and Environment)

Course homepage and information

- **Course homepage**
  - `www.math.chalmers.se/Math/Grundutb/CTH/mve165/1415`
  - Details, information on assignmnents and computer exercises, deadlines, lecture notes, exercises etc
  - Will be updated with new information every week
- **PingPong**
  - `https://pingpong.chalmers.se`
  - Software download (AMPL & CPLEX)
  - Hand-in of assignments

Organization

- **Lectures** – mathematical optimization theory
- **Computer exercise** – learn how to use software solvers
- **Guest lectures** – applications of optimization $\Rightarrow$ assignments
- **Assignments** – modelling, use solvers, analyze solutions, write reports, opposition & oral presentation
- *Assignment work should be done in groups of $\leq$ two persons*
- Define your project groups on the PingPong page of MVE165/MMG631
- The name of the project group must be: *"FirstName1 Surname1 - FirstName2 Surname2"*
- GU and PhD students not having PingPong-entries: *sign a list*
- Computers are reserved most Wednesdays at 13.15–15.00 (room T7203) and 15.15–17.00 (room FT4011) (check the TimeEdit schedule). Teachers will be present only when indicated in the course plan on the home page.

Software

- **A computer exercise on linear optimization and software** is found on the homepage. You are highly recommended to perform it to prepare for the assignments.
- **AMPL-packages** to install on your own computer (linux, mac, windows) is available via Ping-pong (from Tuesday afternoon). Read the agreement text! Also presented on Lecture 2.
- **Matlab**
- **A java-applet for learning the branch-and-bound algorithm**

Course evaluation process

- Questionnaires will be sent out to all students registered for the course after the exam week
- The examiner calls the course representatives to the introductory, (recommended) middle, and final meetings
- Course representatives
  - ??
  - ??

Literature

- **Main course book**:
  - English version: Optimization (2010)
  - Swedish version: Optimeringslära (2008)

  by J. Lundgren, M. Rönnqvist, and P. Värbrand. Studentlitteratur.

- **Exercise book**:
  - English version: Optimization Exercises (2010)
  - Swedish version: Optimeringslära Övningsbok (2008)

  by M. Henningsson, J. Lundgren, M. Rönnqvist, and P. Värbrand. Studentlitteratur.

- Cremona/Studentlitteratur/Adlibris/...

- Also some **hand-outs** (denoted in the lecture notes)

Examination requirements

- Perform **three project assignments** in groups of two students
  - For Assignment 3 there will be two alternatives
- **Written reports** of three assignments
- **A written opposition** to another group's report of Assignment 2
- **An oral presentation** of Assignment 3 (week 22)
- **Presence** at one full oral presentation session
- *To be able to receive a grade higher than 3 or G, the written reports and opposition as well as the oral presentation must be of high quality. Students aiming at grade 4, 5, or VG must also pass an oral exam*

Overview of the lectures and course contents

Mathematical subjects

- Linear optimization models, modelling, theory, solution methods, and sensitivity analysis
- Discrete optimization models, properties, and solution methods
- Optimization problems that can be modelled as flows in networks, theory, and solution methods
- Multi-objective optimization
- Overview of non-linear optimization models, properties, and solution methods
- Mixes of the above

Activities

- Applications of optimization
- Mathematical modelling
- Theory – mathematical properties of models
- Solution techniques – algorithms
- Software solvers

Optimization: "Do something as good as possible"

- **Something:** Which are the decision alternatives?
  - $x_{ij} = \begin{cases} 1 & \text{if customer } j \text{ is visited } \textit{directly} \text{ after customer } i \\ 0 & \text{else} \end{cases}$
  - $y_{ik} = \begin{cases} 1 & \text{if customer } i \text{ is visited by vehicle } k \\ 0 & \text{else} \end{cases}$
- **Possible:** What restrictions are there?
  - Each customer should be visited exactly once
  - Time windows, transport needs and capacity, pick up at one customer – deliver at another, different types of vehicles, ...
- **Good:** What is a relevant optimization criterion?
  - Minimize the total distance / travel time / emissions / waiting time / ...

The classical travelling salesperson problem

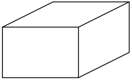
- Variants of routing problems: e.g., refrigerated goods, transportation service for disabled persons, school buses, ...

Examples of application areas

- **Logistics: production and transport**
  - Optimize routes for transports, snow removal, school buses, ...
  - Location of stores
  - Planning of wood cut and transports
  - Packing of containers
  - Production planning and scheduling
- **Energy**
  - Energy production planning
  - Investment in energy production technology
  - Location of power plants and infrastructure
- **Finance**
  - Financial risk management
  - Portfolio optimization
  - Investment planning
- **Medicine**
  - Compute radiation directions/intensities for cancer treatment
  - Reconstruct images from x-ray measurements

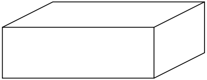
A manufacturing example: Produce tables and chairs from two types of blocks

Small block



$\times 8$

Large block

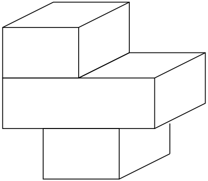


$\times 6$

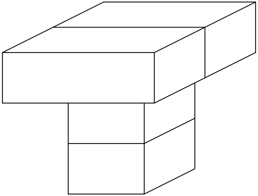
$\Rightarrow$

$\Rightarrow$

Chair



Table



A manufacturing example, continued

- A chair is assembled from one large and two small blocks
- A table is assembled from two blocks of each size
- Only 6 large and 8 small blocks are avaliable
- A table is sold at a revenue of 1600:-
- A chair is sold at a revenue of 1000:-
- Assume that all items produced can be sold and determine an optimal production plan

A mathematical optimization model

- **Something** – What decision alternatives? $\Rightarrow$ Variables

$x_1 =$ number of tables produced and sold

$x_2 =$ number of chairs produced and sold

- **Possible** – What restrictions? $\Rightarrow$ Constraints
  - Maximum supply of large blocks: 6

$$\boxed{2x_1 + x_2 \leq 6}$$

  - Maximum supply of small blocks: 8

$$\boxed{2x_1 + 2x_2 \leq 8}$$

  - Physical restrictions (also: x_1, x_2 integral)

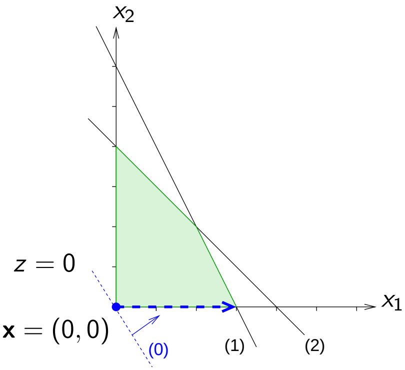
$$\boxed{x_1, x_2 \geq 0}$$

- **Good** – Relevant optimization criterion? $\Rightarrow$ Objective function
  - Maximize the total revenue

$$\boxed{1600x_1 + 1000x_2 \to \max}$$

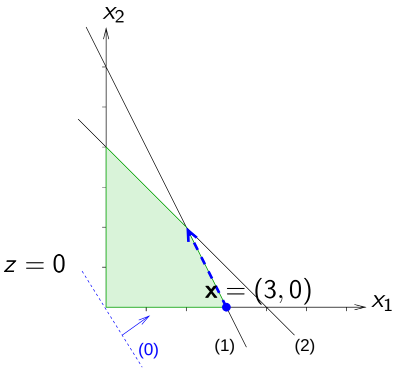
Solve the model using LEGO and marginal values

- Start at no production:
  $x_1 = x_2 = 0$
  Use the "best marginal profit" to choose the item to produce
  - x_1 has the highest marginal profit (1600:-/table)
    $\Rightarrow$ produce as many tables as possible
  - At $x_1 = 3$: no more large blocks left



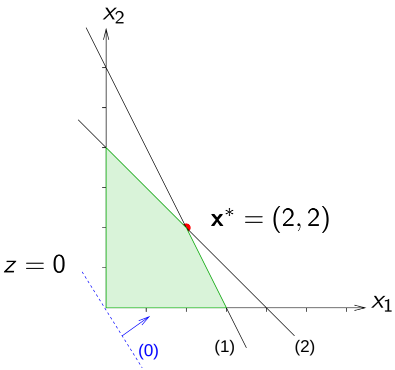
Solve the model using LEGO and marginal values

- The marginal value of x_2 is now 200:- since taking apart one table (-1600:-) yields two chairs ($+2000$:-) $\Rightarrow$ 400:-/2 chairs
  - Increase x_2 maximally $\Rightarrow$ decrease x_1
  - At $x_1 = x_2 = 2$: no more small blocks



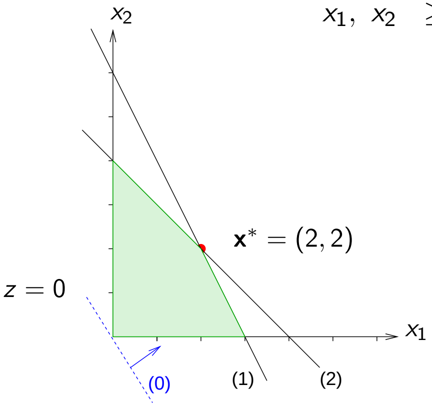
Solve the model using LEGO and marginal values

- The marginal value of x_1 is negative (to build one more table one has to take apart two chairs $\Rightarrow$ -400:-)
  The marginal value of x_2 is -600:- (to build one more chair one table must be taken apart)
  $\Longrightarrow$ Optimal solution: $x_1 = x_2 = 2$



Geometric solution of the model

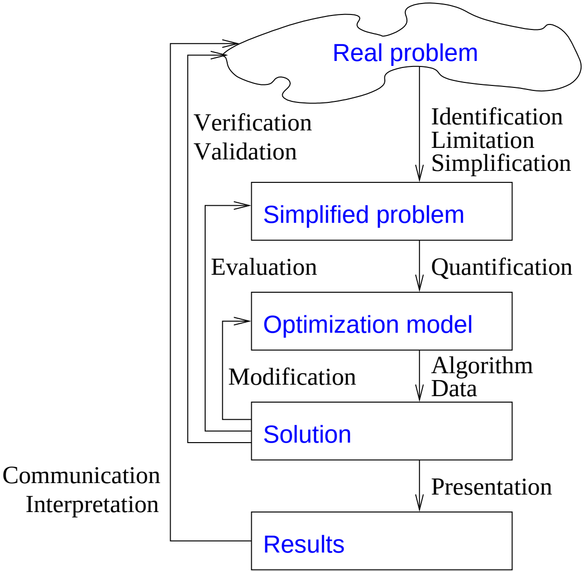
$$
\begin{array}{llrcrcrl}
\text{maximize} & z = & 1600x_1 & + & 1000x_2 & & & (0) \\
\text{subject to} & & 2x_1 & + & x_2 & \leq & 6 & (1) \\
 & & 2x_1 & + & 2x_2 & \leq & 8 & (2) \\
 & & & & x_1,\ x_2 & \geq & 0 &
\end{array}
$$



Operations Research (OR) (Swedish: Operationsanalys)

- Scientific view on problem solving regarding complex systems
- *"OR means to prepare a foundation for rational decisions by utilizing systematic scientific methods and — when possible and meaningful — by utilizing quantitative models"*
- The problem is considered as a system of components which cooperate and influence each other
- The activities studied are described by models, used to
  - better understand the depicted system,
  - understand the consequences of different decisions, and
  - choose the "best" alternative due to some criterion.

The process of optimization

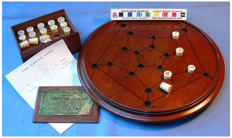


History of Operations Research

- During world war II decision problems became systematically treated: Operations Research
- After the war: use of operations research for civil operations
- The ideas spread to many countries
- Early operations research include inventory planning

A few moments in optimization history

- Euler (1735): Seven bridges of Köningsberg (graph theory)
- Gauss (1777–1855): 1st optimization technique – steepest descent
- W.R. Hamilton (1857): "icosian game" $\Rightarrow$ the travelling salesperson problem (Hamilton cycle)



- L.V. Kantorovich (1939): A linear model for optimization of plywood manufactoring and an algorithm for its solution
- George B. Dantzig (1947): Linear programming – the simplex algorithm (exponential time)
  - Program $\Leftrightarrow$ military training and logistics schedules
- Kantorovich and Koopmans (1975): Nobel Memorial Prize in Economic Sciences
- N. Karmarkar (1984) algorithm for linear programming (polynomial time)

Optimization modelling: Electric power capacity expansion

- An electric utility will install two generators ($j = 1, 2$) with different fixed and operating costs, to meet the demand within its service region.
- Each day is divided into three *parts* ($i = 1, 2, 3$) of equal duration, during which the demand, d_i, takes a *base*, *medium*, or *peak* value, respectively.
- The *fixed cost* per unit capacity of generator j is amortized over its lifetime and amounts to c_j per day.
- The *operating cost* of generator j during the ith part of the day is f_{ij}.
- The *availability* of generator j is $a_j \in [0, 1]$, $j = 1, 2$.
- If the demand during the ith part of the day cannot be served due to lack of capacity, *additional capacity* must be purchased at a cost of g_i per unit.
- The *capacity* of each generator j is required to be at least b_j.

Define the decision variables

- Let the variables x_j, $j = 1, 2$, represent the installed capacity of generator j.
- Let the variables y_{ij} denote the operating level of generator j during the ith part of the day.
- Let the variable w_i denote the capacity that needs to be purchased, in order to satisfy unmet demand during the ith part of the day.
- We interpret availability to mean that the operating level of generator j, at any given time is *at most* $a_j x_j$.

General mathematical optimization models

-

$$\left[\begin{array}{lll} \text{minimize or maximize} & f(x_1,\ldots,x_n) & \\ \text{subject to} & g_i(x_1,\ldots,x_n) \left\{\begin{array}{c}\leq \\ = \\ \geq\end{array}\right\} b_i, & i=1,\ldots,m \end{array}\right]$$

- $x_1, \ldots, x_n$ are the decision variables
- f and $g_1, \ldots, g_m$ are given functions of the decision variables
- $b_1, \ldots, b_m$ are specified constant parameters
- The functions can be nonlinear, e.g. quadratic, exponential, logarithmic, non-analytic, ...
- In general, linear forms are more tractable than non-linear

Linear optimization models (programs)

- The capacity expansion model is a linear program (LP), i.e., all relations are described by linear forms

- A general linear program:

$$\left[\begin{array}{lcccl} \text{min or max} & c_1x_1 + \ldots + c_nx_n & & & \\ \text{subject to} & a_{i1}x_1 + \ldots + a_{in}x_n & \left\{\begin{array}{c}\leq\\=\\\geq\end{array}\right\} & b_i, & i = 1, \ldots, m \\ & x_j & \geq & 0, & j = 1, \ldots, n \end{array}\right]$$

- The non-negativity constraints on x_j, $j = 1, \ldots, n$ are not necessary, but usually assumed (reformulation always possible)

Discrete/integer/binary modelling

- A variable is called *discrete* if it can take only a countable set of values, e.g.,
  - Continuous variable: $x \in [0, 8] \iff 0 \leq x \leq 8$
  - Discrete variable: $x \in \{0, 4.4, 5.2, 8.0\}$
  - *Integer* variable: $x \in \{0, 1, 4, 5, 8\}$
- A *binary* variable can only take the values 0 or 1, i.e., all or nothing

  E.g., a wind-mill can produce electricity only if it is built
  - Let $y = 1$ if the mill is built, otherwise $y = 0$
  - Capacity of a mill: C
  - Production $x \leq Cy$ (also limited by wind force etc.)
- In general, models with only continuous variables are more tractable than models with integrality/discrete requirements on the variables, but exceptions exist! More about this later.