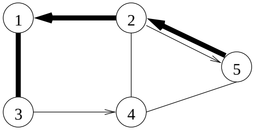
MVE165/MMG631
Linear and integer optimization with applications
Lecture 11
Shortest paths and network flows; linear programming formulations

Ann-Brith Strömberg

2015–05–12

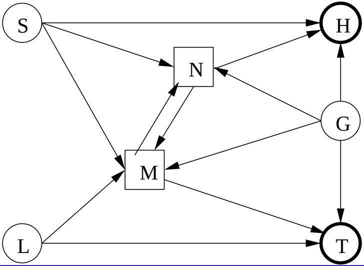
Flows in networks, in particular shortest paths

A path from node 5 to node 3



A flow network

- Supply nodes: S, G, L
- Demand nodes: H, T
- Storage (intermediate): M, N
- Limited capacities on links
- Minimize costs for transport and storage



Many different problems can be formulated as graph or network flow models

- Find the total capacity of a given water pipeline network
- Find a time schedule (starting and completion times) for the activities in a project
- How much goods should be transported from each supplier to each point of demand in a transportation system, and which links should be used to what extent

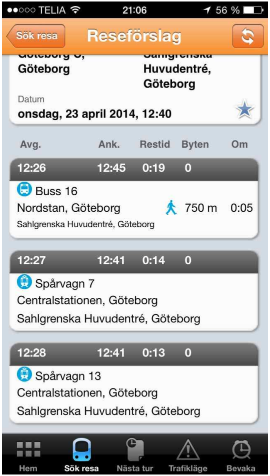
A useful application

Question:

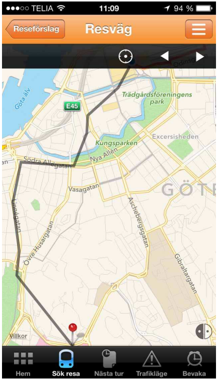
In terms of networks

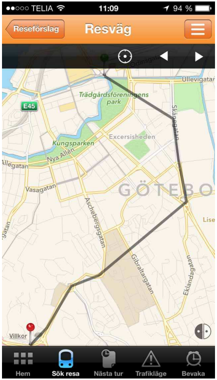
- What question do we ask?
- Discuss with your neigbour!
- Suggestions?

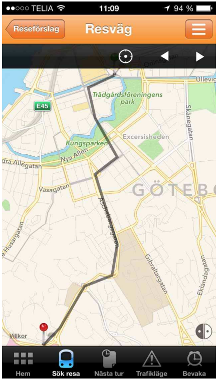
Answer:

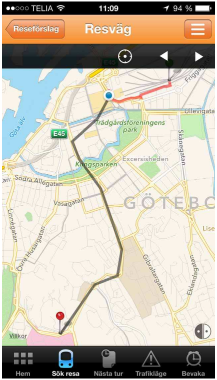


The shortest path problem: a useful application







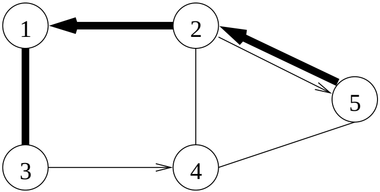


A number of “short” (or fast) paths that

- depart at the earliest “now”, and
- arrive at the latest “around 12:40”

Shortest path problem—properties & solution

- What properties of the problem can we utilize to construct an efficient solution method for the shortest path problem?



- Discuss
- *... construction on the board ...*

Solving the problem ...

- How long is the shortest path from 1 to 6? Why?
- Discuss
- How can we find this path, using the "spatial" properties of the network?
- Discuss
- *... utilize construction on the board ...*

A mathematical model

Let y_i = length of the shortest path from node 1 to node i

- "Stretch the threads" between the nodes 1 and 6 $\Leftrightarrow$ maximize the difference of the "potentials" y_6 and y_1:

$$(y_6 - y_1) \longrightarrow \max$$

- The threads are not elastic:

$$\begin{aligned} y_2 - y_1 &\leq 4 \qquad & y_4 - y_2 &\leq 3 \qquad & y_5 - y_4 &\leq 4 \\ y_3 - y_1 &\leq 2 & y_4 - y_3 &\leq 2 & y_6 - y_4 &\leq 4 \\ y_2 - y_3 &\leq 3 & y_5 - y_2 &\leq 4 & y_6 - y_5 &\leq 1 \end{aligned}$$

- A system of nine inequalities (not equations) and six unknowns, as well as an objective function to be maximized

Another mathematical model—based on flows

Send one unit of flow along the shortest path from node 1 to node 6

- Let
$$x_{ij} = \begin{cases} 1 & \text{if link } (i,j) \text{ is in the shortest path from 1 to 6} \\ 0 & \text{otherwise} \end{cases}$$
- Objective:
$$(4x_{12} + 2x_{13} + 3x_{32} + 3x_{24} + 2x_{34} + 4x_{25} + 4x_{45} + 4x_{46} + 1x_{56}) \to \min$$
- Node balance (any flow that enters a node must also leave it)

$$\begin{array}{rrrrrrrrrcr}
-x_{12} & -x_{13} & & & & & & & & = & -1 \\
+x_{12} & & +x_{32} & -x_{24} & & -x_{25} & & & & = & 0 \\
& +x_{13} & -x_{32} & & -x_{34} & & & & & = & 0 \\
& & & +x_{24} & +x_{34} & & -x_{45} & -x_{46} & & = & 0 \\
& & & & & +x_{25} & +x_{45} & & -x_{56} & = & 0 \\
& & & & & & & +x_{46} & +x_{56} & = & 1 \\
x_{12}, & x_{13}, & x_{32}, & x_{24}, & x_{34}, & x_{25}, & x_{45}, & x_{46}, & x_{56} & \geq & 0
\end{array}$$

A mathematical model

The optimal solution

- $y_1^* = 0,\ y_2^* = 4,\ y_3^* = 2,\ y_4^* = 4,\ y_5^* = 8,\ y_6^* = 8$
- $\Leftrightarrow$ maximize the difference of the potentials:
$$(y_6^* - y_1^*) = 8$$
- Fulfilment of the constraints:
$$\begin{array}{lll} y_2^* - y_1^* = 4 = 4 & y_4^* - y_2^* = 0 < 3 & y_5^* - y_4^* = 4 = 4 \\ y_3^* - y_1^* = 2 = 2 & y_4^* - y_3^* = 2 = 2 & y_6^* - y_4^* = 4 = 4 \\ y_2^* - y_3^* = 2 < 3 & y_5^* - y_2^* = 4 = 4 & y_6^* - y_5^* = 0 < 1 \end{array}$$
- The optimal solution to the flow model:
$$x_{13}^* = x_{34}^* = x_{46}^* = 1$$
$$x_{12}^* = x_{32}^* = x_{24}^* = x_{25}^* = x_{45}^* = x_{56}^* = 0$$

[Illustrate the complementarity]

A linear programming formulation: shortest path from node $s \in N$ to node $t \in N$ in a directed graph $G = (N, A, \mathbf{d})$

- For each arc $(i,j) \in A$, let x_{ij} be the flow on the arc
- *Flow balance in each node $k \in N$*
- $x_{ij} = 1$ if arc (i,j) is in the shortest path and $x_{ij} = 0$ otherwise

Linear programming formulation (assume $d_{ij} \geq 0$):

$$
\begin{array}{rrcl}
\min & \displaystyle\sum_{(i,j)\in A} d_{ij} x_{ij}, & & \\
\text{s.t.} & \displaystyle\sum_{i:(i,k)\in A} x_{ik} - \sum_{j:(k,j)\in A} x_{kj} & = & \begin{cases} -1, & k = s, \\ 1, & k = t, \\ 0, & k \in N \setminus \{s,t\}, \end{cases} \\
& x_{ij} & \geq & 0, \quad (i,j) \in A
\end{array}
$$

Linear programming dual:

$$
\begin{array}{rrcll}
\max & y_t - y_s, & & & \\
\text{s.t.} & y_j - y_i & \leq & d_{ij}, & (i,j) \in A \\
& y_k & & \text{free}, & k \in N
\end{array}
$$

The shortest path problem (Ch. 8.4)

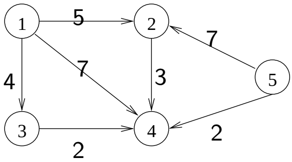
- Given: a network/graph of nodes N, (directed) arcs A, and arc distances d_{ij}, $(i, j) \in A$
- Denoted $G = (N, A, \mathbf{d})$
- Find the shortest path from a source node ($s \in N$) to a destination node ($t \in N$)

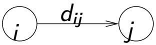
Principle of optimality formulated by Bellman's equations (Ch. 8.4.1)

- In a graph with *no negative cycles*, optimal paths have optimal subpaths
- A shortest path from node s node to t that passes through node k contains a shortest path from node s node to k
- Let y_j denote the length of the shortest path from node s to j

Bellman's equations:

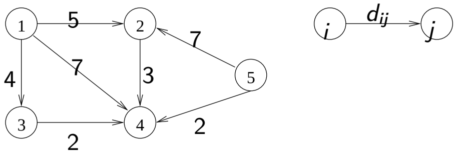
- $y_s = 0$
- $y_j = \min\limits_i \left\{ y_i + d_{ij} : \text{ arc/edge } (i,j) \text{ exists } \right\}$ for all $j \neq s$





Solution method I: Bellman's equations

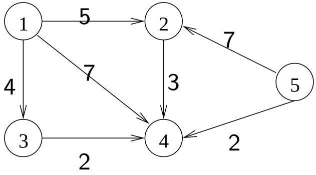
- If the graph is directed without cycles: solve Bellman's equations in topological order
- Shortest path from node 1 to each of the other nodes (1,5,2,3,4):
  - $y_1 := 0$
  - $y_5 := \min\{\infty\} = \min\{\infty\} = \infty$
  - $y_2 := \min\{\infty; y_1 + d_{12}; y_5 + d_{52}\} = \min\{\infty; 0 + 5; \infty\} = 5$
  - $y_3 := \min\{\infty; y_1 + d_{13}\} = \min\{\infty; 0 + 4\} = 4$
  - $y_4 := \min\{\infty; y_1 + d_{14}; y_2 + d_{24}; y_3 + d_{34}; y_5 + d_{54}\} = \min\{\infty; 0 + 7; 5 + 3; 4 + 2; \infty + 2\} = 6$

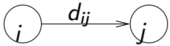


- $y_1^* = 0,\ y_2^* = 5,\ y_3^* = 4,\ y_4^* = 6,\ y_5^* = \infty$

Solution method II: Dijkstra's algorithm

- The graph may contain cycles but all edge costs must be non-negative (i.e., $d_{ij} \geq 0$)





- *Solve the example on the board*

Algorithms for the shortest path problem: Dijkstra (Ch.8.4.2)

- Find the shortest path between node s and node i when all arcs distances are non-negative (cycles may exist)
- N = set of all nodes; source node $s \in N$
- d_{ij} = distance on link from i to j for all $i, j \in N$
- $d_{ij} := \infty$ if no direct link from i to j

Dijkstra's shortest path algorithm

Step 0: $S := \{s\}$, $\bar{S} := N \setminus \{s\}$, and $y_i := d_{si}$, $i \in N$

Step 1:

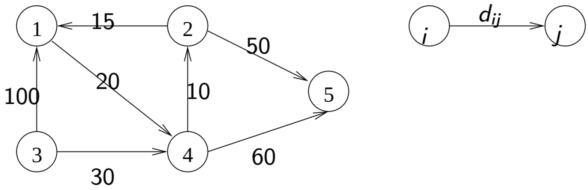
(a) If $\bar{S} = \emptyset$, stop. Otherwise, find node $j \in \bar{S}$ such that $y_j = \min_{i \in \bar{S}} \{y_i\}$. Set $S := S \cup \{j\}$ and $\bar{S} := \bar{S} \setminus \{j\}$

(b) For all $k \in \bar{S}$ and $i \in S$: If $y_k > y_i + d_{ik}$ set $y_k := y_i + d_{ik}$ and $pred(k) := i$. Repeat from (a).

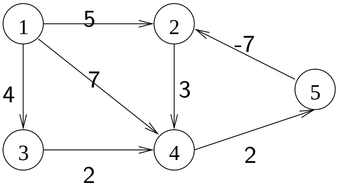
- The vector *pred* keeps track of the predecessors
- Dijkstra's algorithm actually finds shortest paths from the source to *all* others nodes (this is not formulated in the LP)

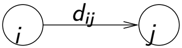
Example: Dijkstra's algorithm

Find the shortest path from node 1 to all other nodes (Homework)



Negative lengths of edges and negative cycles





- Negative length of edges: extend Dijkstra's algorithm according to "move nodes back from S to $\bar{S}$" (Ford's algorithm)
- There may be a cycle of *negative* total length

$\Rightarrow$ "Length" of the shortest path $\rightarrow -\infty$

$\Rightarrow$ Ford's algorithm *either* finds a shortest path *or* detects a cycle with a negative total length

Algorithms for the shortest path problem: Floyd–Warshall (Ch. 8.4.2)

- Computes shortest paths between each pair of nodes
- Negative distances are allowed; negative cycles are detected
- Idea: Three nodes i, k, j and distances d_{ik}, d_{kj}, and d_{ij}
- $i \to k \to j$ is a short-cut if $d_{ik} + d_{kj} < d_{ij}$
- In each iteration $1 \ldots k$, check whether d_{ij} can be improved by using the short-cut via k
- Administration of the algorithm: Maintain two matrices per iteration: $D[k]$ for the distances and $pred[k]$ to keep track of the predecessor of each node

Floyd–Warshall's algorithm

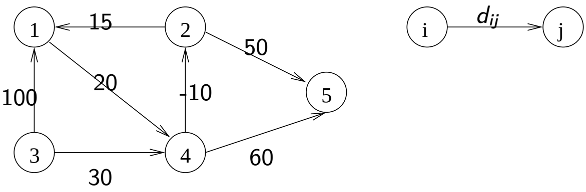
Floyd–Warshall's algorithm

Step 0: Initialize $D[0]$ and $pred[0]$

Step k:

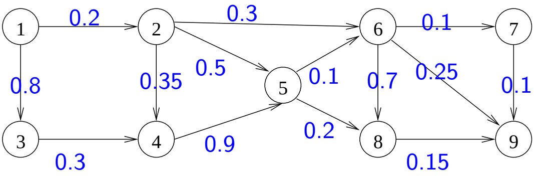
- $D[k] := D[k-1]$, $pred[k] := pred[k-1]$
  For each element d_{ij} in $D[k]$:
  If $d_{ik} + d_{kj} < d_{ij}$, set $d_{ij} := d_{ik} + d_{kj}$ and $pred_{ij}[k] := k$
  Set $k := k+1$
  If $k > n$ stop, else repeat Step k

Find the shortest path from node 3 to all other nodes



Example: Most reliable route

- Mr Q drives to work daily
- All road links he can choose for a path to work are patrolled by the police
- It is possible to assign a probability $p_{ij} \in [0, 1]$ of *not* being stopped by the police on link (i, j)
- Mr Q wants to find the "shortest" (safest?) path in the sense that the probability of being stopped is as low as possible
- maximize $Prob$(not being stopped)



- Ex. $1 \to 4$: $\max\{p_{12}p_{24}; p_{13}p_{34}\} = \max\{0.2 \cdot 0.35; 0.8 \cdot 0.3\}$
- Note: This version *cannot* be formulated as a linear program

Alternative objectives $\Rightarrow$ Variants of Bellman's equations

Most reliable path (failure probability $p_{ij} \in [0,1]$ for arc (i,j)):

- $y_s = 1$
- $y_j = \max_i \left\{ y_i \cdot p_{ij} : \text{ arc/edge } (i,j) \text{ exists } \right\}$ for all $j \neq s$

Highest capacity path (capacity $K_{ij} \geq 0$ on arc (i,j)):

- $y_s = \infty$
- $y_j = \max_i \left\{ \min\{y_i; K_{ij}\} : \text{ arc/edge } (i,j) \text{ exists } \right\}, \ j \neq s$