MVE165/MMG631
Linear and integer optimization with applications
Lecture 5
Discrete optimization models and applications; complexity

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Modelling with integer variables (Ch. 13.1)

Variables

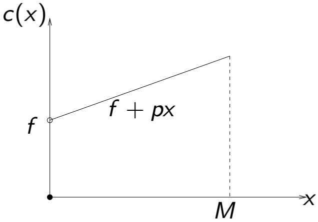
- *Linear programming* (LP) uses continuous variables: $x_{ij} \geq 0$
- *Integer linear programming* (ILP) use also *integer*, *binary*, and *discrete* variables
- If both continuous and integer variables are used in a program, it is called a *mixed integer (linear) program* (MILP)

Constraints

- In an ILP (or MILP) it is possible to model linear constraints, but also logical relations as, e.g. if–then and either–or
- This is done by introducing additional binary variables and additional constraints

Mixed integer modelling—fixed charges

- Send a truck $\Rightarrow$ Start–up cost $f > 0$
- Load bread loafs $\Rightarrow$ cost $p > 0$ per loaf
- $x = \#$ bread loafs to transport from bakery to store



The cost function $c : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is *nonlinear* and *discontinuos*

$$c(x) := \begin{cases} 0 & \text{if} \quad x = 0 \\ f + px & \text{if} \quad 0 < x \leq M \end{cases}$$

Integer linear programming modelling—fixed charges

- Let $y = \#$ trucks to send (here y equals 0 or 1)
- Replace $c(x)$ by $fy + px$
- Constraints: $0 \leq x \leq My$ and $y \in \{0, 1\}$

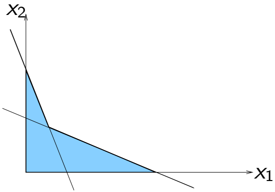
New model:

$$
\begin{array}{lrcl}
\min & fy + px & & \\
\text{s.t.} & x - My & \leq & 0 \\
 & x & \geq & 0 \\
 & y & \in & \{0,1\}
\end{array}
$$

- $y = 0 \quad \Rightarrow \quad x = 0 \quad \Rightarrow \quad fy + px = 0$
- $y = 1 \quad \Rightarrow \quad x \leq M \quad \Rightarrow \quad fy + px = f + px$
- $x > 0 \quad \Rightarrow \quad y = 1 \quad \Rightarrow \quad fy + px = f + px$
- $x = 0 \quad \not\Rightarrow \quad y = 0$ But: Minimization will push y to zero!

Discrete alternatives

- Suppose:
  either $x_1 + 2x_2 \leq 4$ **or** $5x_1 + 3x_2 \leq 10$,
  and $x_1, x_2 \geq 0$ must hold
- **Not** a convex set



Let $M \gg 1$ and define $y \in \{0, 1\}$

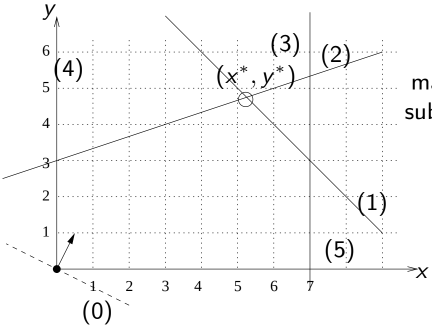
$\Rightarrow$ New constraint set:
$$\left[\begin{array}{rrcl} x_1 + 2x_2 & -My & \leq & 4 \\ 5x_1 + 3x_2 & -M(1-y) & \leq & 10 \\ & y & \in & \{0,1\} \\ x_1, x_2 & & \geq & 0 \end{array}\right]$$

- $y = \begin{cases} 0 & \Rightarrow \quad x_1 + 2x_2 \leq 4 \quad \text{must hold} \\ 1 & \Rightarrow \quad 5x_1 + 3x_2 \leq 10 \quad \text{must hold} \end{cases}$

Exercises: Homework

1. Suppose that you are interested in choosing from a set of investments $\{1, \ldots, 7\}$ using $0-1$ variables. Model the following constraints.
    1. You cannot invest in all of them
    2. You must choose at least one of them
    3. Investment 1 cannot be chosen if investment 3 is chosen
    4. Investment 4 can be chosen only if investment 2 is also chosen
    5. You must choose either both investment 1 and 5 or neither
    6. You must choose either at least one of the investments 1, 2 and 3 or at least two investments from 2, 4, 5 and 6
2. Formulate the following as mixed integer progams
    1. $u = \min\{x_1, x_2\}$, assuming that $0 \leq x_j \leq C$ for $j = 1, 2$
    2. $v = |x_1 - x_2|$ with $0 \leq x_j \leq C$ for $j = 1, 2$
    3. The set $X \setminus \{x^*\}$ where $X = \{x \in Z^n | Ax \leq b\}$ and $x^* \in X$

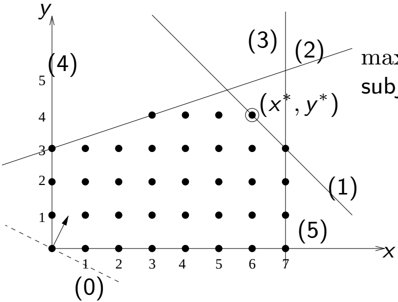
Linear programming: A small example



$$
\begin{array}{lrcrcrl}
\text{maximize} & x & + & 2y & & & (0) \\
\text{subject to} & x & + & y & \leq & 10 & (1) \\
 & -x & + & 3y & \leq & 9 & (2) \\
 & x & & & \leq & 7 & (3) \\
 & & & x, y & \geq & 0 & (4,5)
\end{array}
$$

- Optimal solution: $(x^*, y^*) = (5\frac{1}{4}, 4\frac{3}{4})$
- Optimal objective value: $14\frac{3}{4}$

Integer linear programming: A small example



$$
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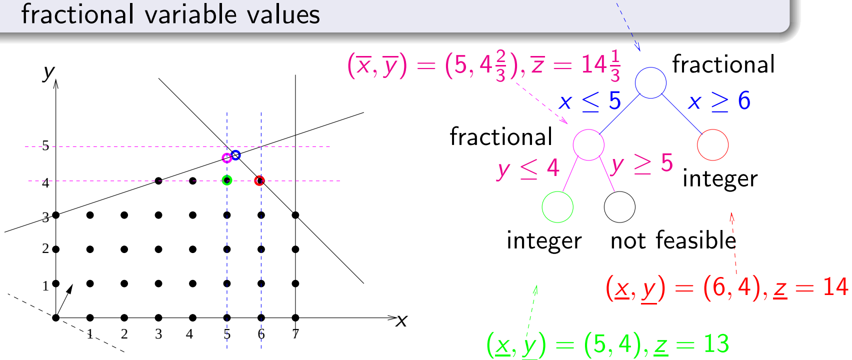
\text{subject to} & x & + & y & \leq & 10 & (1) \\
 & -x & + & 3y & \leq & 9 & (2) \\
 & x & & & \leq & 7 & (3) \\
 & & & x, y & \geq & 0 & (4,5) \\
 & & & x, y & \text{integer} & &
\end{array}
$$

- What if the variables are forced to be integral?
- Optimal solution: $(x^*, y^*) = (6, 4)$
- Optimal objective value: $14 < 14\frac{3}{4}$
- The optimal value decreases (possibly constant) when the variables are restricted to possess only integral values

ILP: Solution by the branch–and–bound algorithm (e.g., Cplex, XpressMP, or GLPK) (Ch. 15.1–15.2)

- Relax integrality requirements ⇒ linear, continuous problem ⇒ $(\overline{x}, \overline{y}) = (5\frac{1}{4}, 4\frac{3}{4}), \overline{z} = 14\frac{3}{4}$
- Search tree: branch over fractional variable values



The knapsack problem—budget constraints (Ch. 13.2)

- Select an optimal collection of objects or investments or projects or ...
  - $c_j =$ benefit of choosing object j, $j = 1, \ldots, n$
- Limits on the budget
  - $a_j =$ cost of object j, $j = 1, \ldots, n$
  - $b =$ total budget

- Variables: $x_j = \begin{cases} 1, & \text{if object } j \text{ is chosen,} \\ 0, & \text{otherwise.} \end{cases} \quad j = 1, \ldots, n$
- Objective function: $\boxed{\max \sum_{j=1}^{n} c_j x_j}$
- Budget constraint: $\boxed{\sum_{j=1}^{n} a_j x_j \leq b}$
- Binary variables: $\boxed{x_j \in \{0, 1\},\ j = 1, \ldots, n}$

Computational complexity (Ch. 2.6)

A small knapsack instance

$$\begin{array}{lrcl} z_1^* = \max & 213x_1 + 1928x_2 + 11111x_3 + 2345x_4 + 9123x_5 & & \\ \text{subject to} & 12223x_1 + 12224x_2 + 36674x_3 + 61119x_4 + 85569x_5 & \leq & 89\,643\,482 \\ & x_1, \dots, x_5 & \geq & 0, \text{integer} \end{array}$$

- Optimal solution $\mathbf{x}^* = (0, 1, 2444, 0, 0)$, $z_1^* = 27\ 157\ 212$
- Cplex finds this solution in 0.015 seconds

The equality version

$$\begin{array}{lrcl} z_2^* = \max & 213x_1 + 1928x_2 + 11111x_3 + 2345x_4 + 9123x_5 & & \\ \text{subject to} & 12223x_1 + 12224x_2 + 36674x_3 + 61119x_4 + 85569x_5 & = & 89\,643\,482 \\ & x_1, \dots, x_5 & \geq & 0, \text{integer} \end{array}$$

- Optimal solution $\mathbf{x}^* = (7334, 0, 0, 0, 0)$, $z_2^* = 1\ 562\ 142$
- Cplex computations interrupted after 1700 sec. ($\approx \frac{1}{2}$ hour)
  - No integer solution found
  - Best upper bound found: 25 821 000
  - 55 863 802 branch–and–bound nodes visited
  - Only *one* feasible solution exists!

Computational complexity

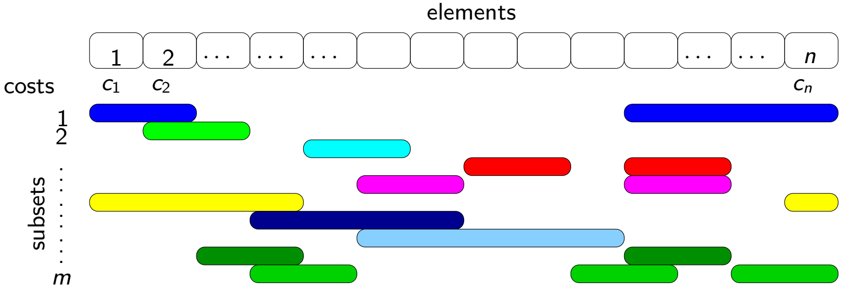
- Mathematical insight yields successful algorithms
- Example: Assignment problem: Assign n persons to n jobs.
- $\#$ feasible solutions: $n!$ $\Rightarrow$ Combinatorial explosion
- An algorithm $\exists$ that solves this problem in time $\mathcal{O}(n^4) \propto n^4$

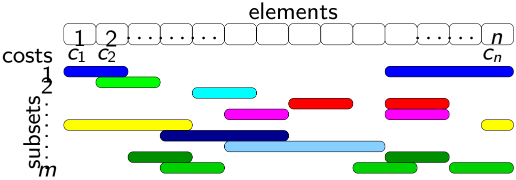
Complete enumeration of all solutions is *not* efficient

n	2	5	8	10	100	1000
$n!$	2	120	40000	3600000	$9.3 \cdot 10^{157}$	$4.0 \cdot 10^{2567}$
2^n	4	32	256	1024	$1.3 \cdot 10^{30}$	$1.1 \cdot 10^{301}$
n^4	16	625	4100	10000	$1.0 \cdot 10^{8}$	$1.0 \cdot 10^{12}$
$n \log n$	0.6	3.5	7.2	10	200	3000

- Binary knapsack: $\mathcal{O}(2^n)$
- Continuous knapsack (sorting of $\frac{c_j}{a_j}$): $\mathcal{O}(n \log n)$

- A number (n) of items and a cost for each item
- A number (m) of subsets of the n items
- Find a selection of the items such that each subset contains at least one selected item and such that the total cost for the selected items is minimized





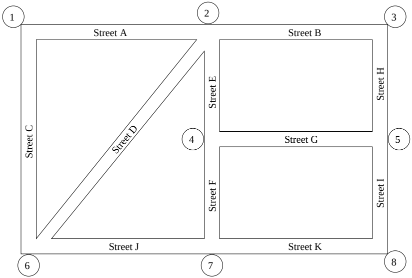
Mathematical formulation

$$\begin{array}{lrcl} \text{min} & \mathbf{c}^{\mathrm{T}}\mathbf{x} & & \\ \text{subject to} & \mathbf{A}\mathbf{x} & \geq & \mathbf{1} \\ & \mathbf{x} & & \text{binary} \end{array}$$

- $\mathbf{c} \in \mathbb{R}^n$ and $\mathbf{1} = (1, \ldots, 1)^{\mathrm{T}} \in \mathbb{R}^m$ are constant vectors
- $\mathbf{A} \in \mathbb{R}^{m \times n}$ is a matrix with entries $a_{ij} \in \{0, 1\}$
- $\mathbf{x} \in \mathbb{R}^n$ is the vector of variables
- Related models: *set partitioning* ($\mathbf{Ax} = \mathbf{1}$), *set packing* ($\mathbf{Ax} \leq \mathbf{1}$)

Example: Installing security telephones

- The road administration wants to install emergency telephones such that each street has access to at least one phone
- It is logical to place the phones at street crossings
- Each crossing has an installation cost: $\mathbf{c} = (2, 2, 3, 4, 3, 2, 2, 1)$
- Find the cheapest selection of crossings to provide all streets with phones



Model

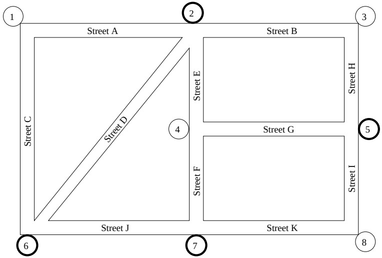
Define variables and constraints

Installing security telephones: Mathematical model

- Binary variables for each crossing: $x_j = 1$ if a phone is installed at j, $x_j = 0$ otherwise.
- For each street, introduce a constraint saying that a phone should be placed at—at least—one of its crossings:

A: $x_1 + x_2 \geq 1$
B: $x_2 + x_3 \geq 1$
C: $x_1 + x_6 \geq 1$
D: $x_2 + x_6 \geq 1$
E: $x_2 + x_4 \geq 1$
F: $x_4 + x_7 \geq 1$

G: $x_4 + x_5 \geq 1$
H: $x_3 + x_5 \geq 1$
I: $x_5 + x_8 \geq 1$
J: $x_6 + x_7 \geq 1$
K: $x_7 + x_8 \geq 1$



- Objective function:
  min $\quad 2x_1 + 2x_2 + 3x_3 + 4x_4 + 3x_5 + 2x_6 + 2x_7 + x_8$
- An optimal solution: $x_2 = x_5 = x_6 = x_7 = 1$, $x_1 = x_3 = x_4 = x_8 = 0$. Objective value: 9.

- Given three telephone companies A, B and, C which charge a fixed start-up price of 16, 25 and, 18, respectively
- For each minute of call-time A, B, and, C charge 0.25, 0.21 and, 0.22
- We want to phone 200 minutes. Which company should we choose?

- x_i = number of minutes called by $i \in \{A, B, C\}$
- Binary variables $y_i = 1$ if $x_i > 0$, $y_i = 0$ otherwise (pay start-up price only if calls are made with company i)

Mathematical model

$$
\begin{array}{lrcll}
\min & 0.25x_1 + 0.21x_2 + 0.22x_3 & + & 16y_1 + 25y_2 + 18y_3 & \\
\text{subject to} & x_1 + x_2 + x_3 & = & 200 & \\
 & 0 \leq x_i & \leq & 200y_i, & i = 1, 2, 3 \\
 & y_i & \in & \{0, 1\}, & i = 1, 2, 3
\end{array}
$$

Process three jobs on one machine

- Each job j has a processing time p_j, a due date d_j, and a penalty cost c_j if the due date is missed
- How should the jobs be scheduled to minimize the total penalty cost?

Job	Processing time (days)	Due date (days)	Late penalty \$/day
1	5	25	19
2	20	22	12
3	15	35	34

HOMEWORK!

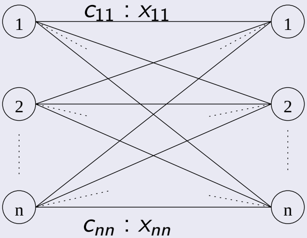
The assignment model (Ch. 13.5)

Assign each task to one resource, and each resource to one task

- A cost c_{ij} for assigning task i to resource j, $i, j \in \{1, \ldots, n\}$
- Variables: $x_{ij} = \begin{cases} 1, & \text{if task } i \text{ is assigned to resource } j \\ 0, & \text{otherwise} \end{cases}$

$$
\begin{array}{lrcll}
\min & \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} & & & \\
\text{subject to} & \displaystyle\sum_{j=1}^{n} x_{ij} & = & 1, & i = 1, \ldots, n \\
& \displaystyle\sum_{i=1}^{n} x_{ij} & = & 1, & j = 1, \ldots, n \\
& x_{ij} & \geq & 0, & i, j = 1, \ldots, n
\end{array}
$$

Choose *one* element from each row and each column

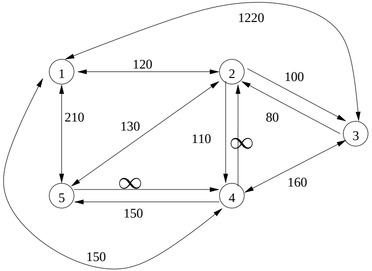


c_{11}	c_{12}	c_{13}					c_{1n}
c_{21}	c_{22}	c_{23}					c_{2n}
c_{31}	c_{32}	c_{33}					c_{3n}
c_{n1}	c_{n2}	c_{n3}					c_{nn}

- This integer linear model has integral extreme points, since it can be formulated as a network flow problem
- Therefore, it can be efficiently solved using specialized (network) linear programming techniques
- Even more efficient special purpose (primal–dual–graph-based) algorithms exist

The travelling salesperson problem (TSP) (Ch. 13.10)

- Given n cities and connections between all cities (distances on each connection)
- Find shortest tour that passes through all the cities



- A problem that is very easy to describe and understand but very difficult to solve (combinatorial explosion)
- ∃ different versions of TSP: Euclidean, metric, symmetric, ...

An ILP formulation of the TSP problem

- Let the distance from city i to city j be d_{ij}
- Introduce a binary variable x_{ij} for each connection
- Let $V = \{1, \ldots, n\}$ denote the set of nodes (cities)

$$
\begin{array}{llcll}
\min & \sum_{i \in V} \sum_{j \in V} d_{ij} x_{ij}, & & & \\
\text{s.t.} & \sum_{j \in V} x_{ij} & = & 1, & i \in V, \quad (1) \\
& \sum_{i \in V} x_{ij} & = & 1, & j \in V, \quad (2) \\
& \sum_{i \in U, j \in V \setminus U} x_{ij} & \geq & 1, & \forall U \subset V : 2 \leq |U| \leq |V| - 2, \quad (3) \\
& x_{ij} & \text{binary} & & i, j \in V \quad (4)
\end{array}
$$

An ILP formulation of the TSP problem

$$
\begin{array}{llcll}
\min & \displaystyle\sum_{i\in V}\sum_{j\in V} d_{ij}x_{ij}, & & & \\
\text{s.t.} & \displaystyle\sum_{j\in V} x_{ij} & = & 1, & i\in V, \qquad (1)\\
 & \displaystyle\sum_{i\in V} x_{ij} & = & 1, & j\in V, \qquad (2)\\
 & \displaystyle\sum_{i\in U, j\in V\setminus U} x_{ij} & \geq & 1, & \forall U\subset V: 2\leq |U|\leq |V|-2, \qquad (3)\\
 & x_{ij} & & \text{binary} & i,j\in V \qquad (4)
\end{array}
$$

- Cf. the assignment problem $\boxed{\text{DRAW GRAPH * 2 !}}$
- Enter and leave each city exactly once ⇔ (1) and (2) $\boxed{\text{DRAW!}}$
- Constraints (3): *subtour elimination* $\boxed{\text{DRAW!}}$
- Alternative formulation of (3): $\boxed{\text{DRAW!}}$
  $\sum_{(i,j)\in U} x_{ij} \leq |U|-1, \quad \forall U\subset V: 2\leq |U|\leq |V|-2$