MVE165/MMG631
Linear and integer optimization with applications
Lecture 13
Multiobjective optimization

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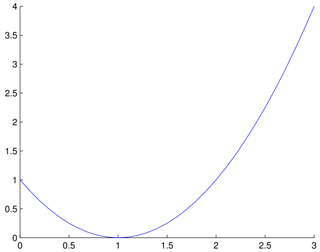
Applied optimization — multiple objectives

- Many practical optimization problems have several objectives which may be in conflict
- Some goals cannot be reduced to a common scale of cost/profit $\Rightarrow$ trade-offs must be addressed
- Examples
  - Financial investments — risk vs. return
  - Engine design — efficiency vs. NO_x vs. soot
  - Wind power production — investment vs. operation (Ass 3a)
  - Electricity generation — costs vs. emissions (Ass 3b)

Literature on multiple objectives' optimization

Copies from the book *Optimization in Operations Research* by R.L. Rardin (1998) pp. 373–387, handed out (on paper, copies kept outside Ann-Brith's office, room MV:L2087)

- Consider the minimization of $f(x) := (x-1)^2$ subject to $0 \leq x \leq 3$
- Optimal solution: $x^* = 1$ (since the function f is convex)



Optimization of multiple objectives

Consider then two objectives

$$\text{minimize } [f_1(x);\ f_2(x)]$$

$$\text{subject to } 0 \leq x \leq 3$$

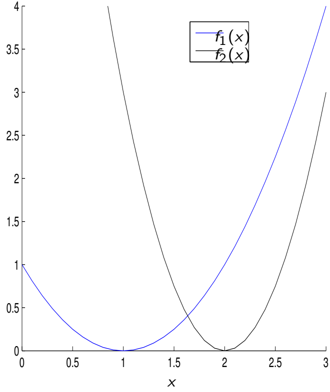
where

$$f_1(x) := (x-1)^2, \quad f_2(x) := 3(x-2)^2$$

- How can an optimal solution by defined?

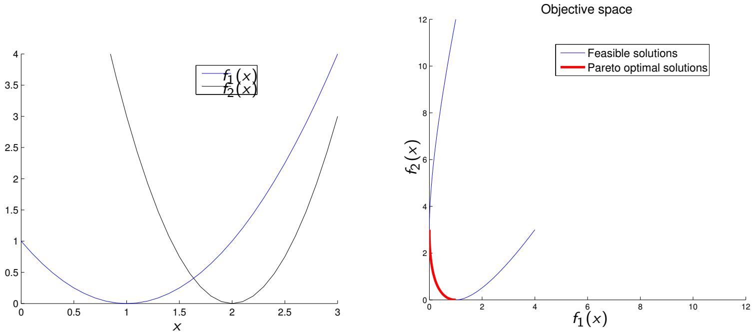
A solution is *Pareto optimal* if *no other* feasible solution has a better value in *all* objectives

- All points $x \in [1, 2]$ are Pareto optimal



Pareto optimal solutions in the objective space

- minimize $[f_1(x);\; f_2(x)]$ subject to $0 \leq x \leq 3$
  where $f_1(x) := (x-1)^2$ and $f_2(x) := 3(x-2)^2$
- A solution is *Pareto optimal* if *no other* feasible solution has a better value in *all* objectives

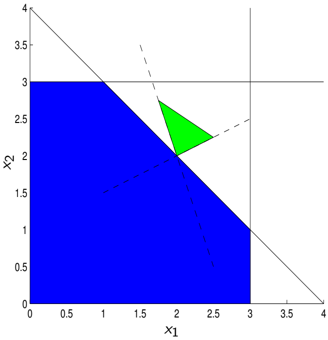


- Pareto optima $\Leftrightarrow$ nondominated points $\Leftrightarrow$ efficient frontier

Efficient points

- Consider a bi-objective linear program:

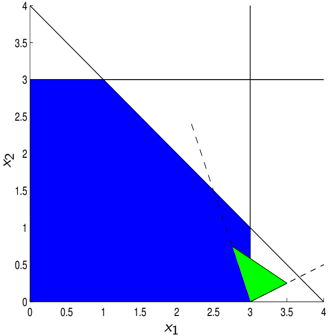
$$\begin{aligned}
\text{maximize} \quad & 3x_1 + x_2 \\
\text{maximize} \quad & -x_1 + 2x_2 \\
\text{subject to} \quad & x_1 + x_2 \leq 4 \\
& 0 \leq x_1 \leq 3 \\
& 0 \leq x_2 \leq 3
\end{aligned}$$



- The solutions in the green cone are *better* than the solution $(2, 2)$ *w.r.t. both objectives*
- The point $x = (2, 2)$ is an *efficient, or non-dominated,* solution

Dominated points

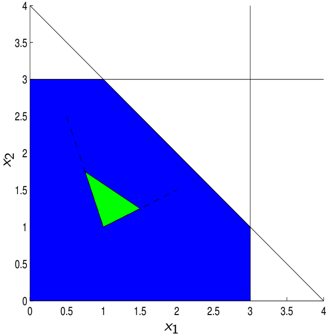
$$\begin{array}{rl} \text{maximize} & 3x_1 + x_2 \\ \text{maximize} & -x_1 + 2x_2 \\ \text{subject to} & x_1 + x_2 \leq 4 \\ & 0 \leq x_1 \leq 3 \\ & 0 \leq x_2 \leq 3 \end{array}$$



- The point $x = (3, 0)$ is *dominated* by the solutions in the green cone
- Feasible solutions exist that are better w.r.t. both objectives

Dominated points

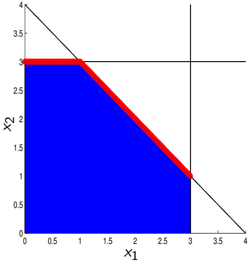
$$
\begin{array}{rl}
\text{maximize} & 3x_1 + x_2 \\
\text{maximize} & -x_1 + 2x_2 \\
\text{subject to} & x_1 + x_2 \leq 4 \\
& 0 \leq x_1 \leq 3 \\
& 0 \leq x_2 \leq 3
\end{array}
$$



- The point $x = (1, 1)$ is dominated by the solutions in the green cone
- Feasible solutions exist that are better w.r.t. both objectives

The efficient frontier—the set of Pareto optimal solutions

$$\begin{array}{ll} \text{maximize} & 3x_1 + x_2 \\ \text{maximize} & -x_1 + 2x_2 \\ \text{subject to} & x_1 + x_2 \leq 4 \\ & 0 \leq x_1 \leq 3 \\ & 0 \leq x_2 \leq 3 \end{array}$$



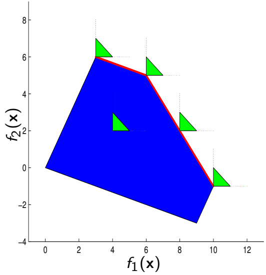
- The set of efficient solutions is given by

$$\left\{ \mathbf{x} \in \Re^2 \,\middle|\, \mathbf{x} = \alpha \begin{pmatrix} 3 \\ 1 \end{pmatrix} + (1-\alpha) \begin{pmatrix} 1 \\ 3 \end{pmatrix}, 0 \leq \alpha \leq 1 \right\} \bigcup$$
$$\left\{ \mathbf{x} \in \Re^2 \,\middle|\, \mathbf{x} = \alpha \begin{pmatrix} 1 \\ 3 \end{pmatrix} + (1-\alpha) \begin{pmatrix} 0 \\ 3 \end{pmatrix}, 0 \leq \alpha \leq 1 \right\}$$

Note that this is *not* a convex set!

The Pareto optimal set in the objective space

$$\begin{array}{ll} \text{maximize} & f_1(\mathbf{x}) := 3x_1 + x_2 \\ \text{maximize} & f_2(\mathbf{x}) := -x_1 + 2x_2 \\ \text{subject to} & x_1 + x_2 \leq 4 \\ & 0 \leq x_1 \leq 3 \\ & 0 \leq x_2 \leq 3 \end{array}$$

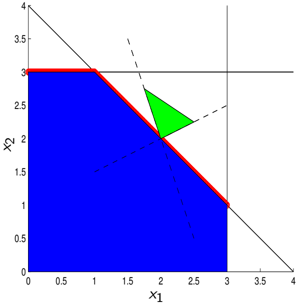


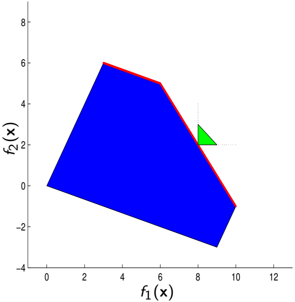
- The set of Pareto optimal objective values is given by

$$\left\{ (f_1, f_2) \in \Re^2 \,\middle|\, \mathbf{f} = \alpha \begin{pmatrix} 10 \\ -1 \end{pmatrix} + (1-\alpha) \begin{pmatrix} 6 \\ 5 \end{pmatrix}, 0 \leq \alpha \leq 1 \right\} \bigcup$$
$$\left\{ (f_1, f_2) \in \Re^2 \,\middle|\, \mathbf{f} = \alpha \begin{pmatrix} 6 \\ 5 \end{pmatrix} + (1-\alpha) \begin{pmatrix} 3 \\ 6 \end{pmatrix}, 0 \leq \alpha \leq 1 \right\}$$

Mapping from the decision space to the objective space

$$\begin{array}{ll} \text{maximize} & [3x_1 + x_2;\ -x_1 + 2x_2] \\ \text{subject to} & x_1 + x_2 \leq 4, \quad 0 \leq x_1 \leq 3, \quad 0 \leq x_2 \leq 3 \end{array}$$





Solutions methods for multiobjective optimization

Construct the efficient frontier by treating one objective as a constraint and optimizing for the other

$$\begin{array}{rl} \text{maximize} & 3x_1 + x_2 \\ \text{subject to} & -x_1 + 2x_2 \geq \varepsilon \\ & x_1 + x_2 \leq 4 \\ & 0 \leq x_1 \leq 3 \\ & 0 \leq x_2 \leq 3 \end{array}$$

- Here, let $\varepsilon \in [-1, 6]$. Why?
- What if the number of objectives is ≥ 3?
- How many single objective linear programs do we have to solve for seven objectives and ten values of ε_k for each objective f_k, $k = 1, \ldots, 7$?
- It is called the *ε-constraints method*

Solution methods: preemptive optimization

- Consider one objective at a time—the most important first
- Solve for the first objective
- Solve for the second objective over the solution set for the first
- Solve for the third objective over the solution set for the second
- ...

- The final solution is an efficient point
- But: Different orderings of the objectives yield different points on the efficient frontier

- Exercise (homework): solve the previous example using preemptive optimization for different orderings of the objective functions

Solution methods: weighted sums of objectives

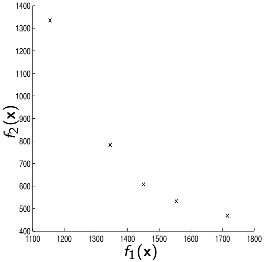
- Give each maximization (minimization) objective a positive (negative) weight
- Solve a single objective maximization problem

$\Rightarrow$ Yields an efficient solution

- Drawback 1: Well spread weights do not necessarily produce solutions that are well spread on the efficient frontier

  Ex: $\left\{\frac{1}{10}, \frac{1}{2}, 1, 2, 10\right\}$

- Drawback 2: If the objectives are *non-concave* (maximization) or if the feasible set is *non-convex*, as, e.g., integrality constrained, then *not all points on the efficient frontier may be possible to detect using weighted sums of objectives*



The efficient frontier in the case of non-convexity

A bi-objective binary linear program

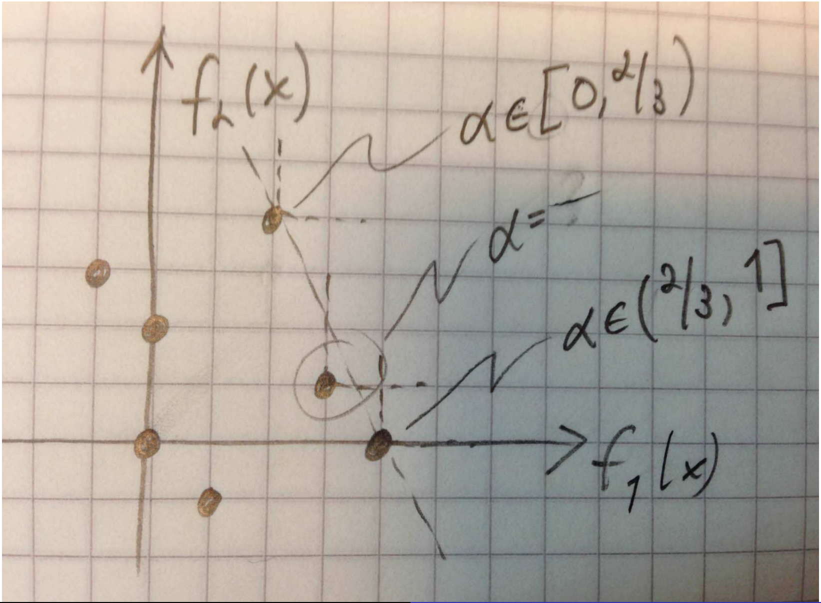
$$\begin{aligned} \text{maximize} \quad & f_1(\mathbf{x}) := 3x_1 + x_2 - x_3 \\ \text{maximize} \quad & f_2(\mathbf{x}) := x_1 - x_2 + 3x_3 \\ \text{subject to} \quad & \mathbf{x} \in X := \left\{ \mathbf{x} \in \mathbb{B}^3 \,\middle|\, x_1 + x_2 + x_3 \leq 2 \right\} \end{aligned}$$

Then,

$$X := \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\},$$

$f_1(X) = \{0, -1, 1, 3, 0, 2, 4\}$ and $f_2(X) = \{0, 3, -1, 1, 2, 4, 0\}$

The efficient frontier in the case of non-convexity



The efficient frontier in the case of non-convexity

Solution by weighted maximization: Let $\alpha \in [0,1]$

$$\begin{aligned}\alpha f_1(\mathbf{x}) + (1-\alpha) f_1(\mathbf{x}) &= \alpha(3x_1 + x_2 - x_3) + (1-\alpha)(x_1 - x_2 + 3x_3) \\ &= (2\alpha+1)x_1 + (2\alpha-1)x_2 + (3-4\alpha)x_3\end{aligned}$$

Resulting binary linear program:

$$\begin{aligned}\text{maximize} \quad & (2\alpha+1)x_1 + (2\alpha-1)x_2 + (3-4\alpha)x_3 \\ \text{subject to} \quad & \mathbf{x} \in X\end{aligned}$$

- $\alpha \in [0, \frac{2}{3}) \implies \mathbf{x}^* = (1,0,1)^{\mathrm{T}}$ & $\mathbf{f}^* = (2,4)^{\mathrm{T}}$
- $\alpha = \frac{2}{3} \Rightarrow \mathbf{x}^* \in \{(1,0,1)^{\mathrm{T}}, (1,1,0)^{\mathrm{T}}\}$ & $\mathbf{f}^* \in \{(2,4)^{\mathrm{T}}, (4,0)^{\mathrm{T}}\}$
- $\alpha \in (\frac{2}{3}, 1] \implies \mathbf{x}^* = (1,1,0)^{\mathrm{T}}$ & $\mathbf{f}^* = (4,0)^{\mathrm{T}}$

- But the Pareto-optimal solution $\mathbf{x}^* = (1,0,0)^{\mathrm{T}}$ with $\mathbf{f}^* = (3,1)^{\mathrm{T}}$ cannot be found using the weighted sums method
- What would the ε-contraints method yield?

Solution methods: ε-constraints

- Consider solving the previous example using the ε-constraint method

The resulting one-objective binary linear program

$$\begin{aligned} \text{maximize} \quad & f_1(\mathbf{x}) := 3x_1 + x_2 - x_3 \\ \text{subject to} \quad & f_2(\mathbf{x}) := x_1 - x_2 + 3x_3 \geq \varepsilon \\ & \mathbf{x} \in X := \left\{ \mathbf{x} \in \mathbb{B}^3 \,\middle|\, x_1 + x_2 + x_3 \leq 2 \right\} \end{aligned}$$

- Then vary ε within relevant bounds (which are these?)

Solution methods: soft constraints

Consider the multiobjective optimization problem to

$$\text{maximize } [f_1(\mathbf{x});\ \ldots;\ f_K(\mathbf{x})] \text{ subject to } \mathbf{x} \in X$$

- Define a *target value* t_k and a *deficiency variable* $d_k \geq 0$ for each objective f_k
- Construct a *soft constraint* for each objective:

$$\text{maximize } f_k(\mathbf{x}) \quad \Rightarrow \quad f_k(\mathbf{x}) + d_k \geq t_k, \quad k = 1, \ldots, K$$

Minimize the sum of deficiencies: (**)

$$\begin{aligned} \text{minimize} \quad & \sum_{k \in K} d_k \\ \text{subject to} \quad & f_k(\mathbf{x}) + d_k \geq t_k, \quad k = 1, \ldots, K \\ & d_k \geq 0, \quad k = 1, \ldots, K \\ & \mathbf{x} \in X \end{aligned}$$

- When is an optimum of (**) an efficient solution? [Draw!!]
- Important: Find first a common scale for f_k, $k = 1, \ldots, K$

Normalizing the objectives

Find a common scale for f_k, $k = 1, \ldots, K$

- Consider the multiobjective optimization problem to

$$\text{maximize } [f_1(\mathbf{x});\ \ldots;\ f_K(\mathbf{x})] \text{ subject to } \mathbf{x} \in X$$

- Define

$$\tilde{f}_k(\mathbf{x}) := \frac{f_k(\mathbf{x}) - f_k^{\min}}{f_k^{\max} - f_k^{\min}}, \quad k = 1, \ldots, K,$$

  where $f_k^{\max} := \max_{\mathbf{x} \in X} \{f_k(\mathbf{x})\}$ and $f_k^{\min} := \min_{\mathbf{x} \in X} \{f_k(\mathbf{x})\}$

- Then, $\tilde{f}_k(\mathbf{x}) \in [0, 1]$ for all $\mathbf{x} \in X$, so that the functions $\tilde{f}_k$ can be compared on a common scale