MVE165/MMG631
Linear and Integer Optimization with Applications
Lecture 3
Extreme points of convex polyhedra; reformulations; basic feasible solutions; the simplex method

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Course evaluation

- The first meeting was held on Friday, March 24 at 9.30.
- The second meeting will be held during week 17 (April, 24–28)
- Notes will be published in the course's PingPong event
- Any voluntary representative from GU is also welcome! Anyone?

Contact any student representative to present your opinion:

- Arvid Bjurklint (TKTEM)
- Frida Eriksson (TKTEM)
- Oskar Holmstedt (TKTEM)
- Stefanus Ivarsson Bergenhem (MPSYS)

Linear programs, convex polyhedra, extreme points

A linear optimization model – a linear program

$$\begin{array}{lll} \textit{minimize} & z = \displaystyle\sum_{j=1}^{n} c_j x_j & \\ \textit{subject to} & \displaystyle\sum_{j=1}^{n} a_{ij} x_j \leq b_i, & i = 1, \dots, m \\ & x_j \geq 0, & j = 1, \dots, n \end{array}$$

c_j, a_{ij}, b_i: constant parameters

In vector notation

$$\begin{array}{ll} \textit{min} & z = \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \textit{s.t.} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}^n \end{array}$$

$\mathbf{c}, \mathbf{x} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{A} \in \mathbb{R}^{m \times n}$

The feasible region is a polyhedron, $X \subset \mathbb{R}^n_+$

$$X := \left\{ \mathbf{x} \geq \mathbf{0}^n \;\middle|\; \sum_{j=1}^{n} a_{ij} x_j \leq b_i, i = 1, \dots, m \right\} = \{ \mathbf{x} \geq \mathbf{0}^n \mid \mathbf{A}\mathbf{x} \leq \mathbf{b} \}$$

Linear programs, convex polyhedra and extreme points (Ch. 4.1)

Definition (Convex combination)

A *convex combination* of the points $\mathbf{x}^p$, $p = 1, \ldots, P$, is a point $\mathbf{x}$ that can be expressed as

$$\mathbf{x} = \sum_{p=1}^{P} \lambda_p \mathbf{x}^p; \qquad \sum_{p=1}^{P} \lambda_p = 1; \qquad \lambda_p \geq 0, \quad p = 1, \ldots, P$$

[Draw on the board]

Linear programs, convex polyhedra and extreme points (Ch. 4.1)

Intersection of linear constraints form a convex set

The feasible region of a linear program is a *convex set*, since for any two feasible points $\mathbf{x}^1$ and $\mathbf{x}^2$ and any $\lambda \in [0,1]$ it holds that

$$\begin{aligned}\sum_{j=1}^{n} a_{ij}\left(\lambda x_j^1 + (1-\lambda)x_j^2\right) &= \lambda \sum_{j=1}^{n} a_{ij}x_j^1 + (1-\lambda)\sum_{j=1}^{n} a_{ij}x_j^2 \\ &\leq \lambda b_i + (1-\lambda)b_i \\ &= b_i, \qquad\qquad i = 1,\dots,m\end{aligned}$$

and

$$\lambda x_j^1 + (1-\lambda)x_j^2 \geq 0, \qquad\qquad j = 1,\dots,n$$

[DRAW ON THE BOARD]

Linear programs, convex polyhedra and extreme points (Ch. 4.1)

Definition (Extreme point (Def. 4.2))

The point $\mathbf{x}^k$ is an *extreme point* of the polyhedron X if $\mathbf{x}^k \in X$ and it is *not* possible to express $\mathbf{x}^k$ as a *strict convex combination* of two distinct points in X.

I.e: Given $\mathbf{x}^1 \in X$, $\mathbf{x}^2 \in X$, and $0 < \lambda < 1$, it holds that $\mathbf{x}^k = \lambda \mathbf{x}^1 + (1-\lambda)\mathbf{x}^2$ only if $\mathbf{x}^k = \mathbf{x}^1 = \mathbf{x}^2$ hold.

[DRAW ON THE BOARD]

Theorem (Optimal solution in an extreme point (Th. 4.2))

Assume that the feasible region $X = \{\mathbf{x} \geq \mathbf{0}^n \mid \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$ *is non-empty and bounded. Then, the minimum value of the objective* $\mathbf{c}^{\mathrm{T}}\mathbf{x}$ *is attained at (at least) one extreme point* $\mathbf{x}^k$ *of* X*.*

A general linear program – notation

Definition (Notation of linear programs)

$$\text{minimize or maximize} \quad c_1x_1 + \ldots + c_nx_n$$

$$\text{subject to} \quad a_{i1}x_1 + \ldots + a_{in}x_n \left\{ \begin{array}{c} \leq \\ = \\ \geq \end{array} \right\} b_i, \quad i = 1, \ldots, m$$

$$x_j \left\{ \begin{array}{l} \leq 0 \\ \text{unrestricted in sign} \\ \geq 0 \end{array} \right\}, \quad j = 1, \ldots, n$$

The blue notation corresponds to the *standard form*

The standard form and the simplex method for linear programs (Ch. 4.2)

- Every linear program can be reformulated such that:
  - all constraints are expressed as *equalities* with *non-negative right hand sides*
  - all variables involved are restricted to be *non-negative*
- Referred to as the *standard form*
- These requirements streamline the calculations of the *simplex method*
- Software solvers (e.g., Cplex, GLPK, Clp, Gurobi, SCIP) handle also inequality constraints and unrestricted variables – the reformulations are made automatically

The simplex method—standard form reformulations

- Slack variables:

$$\left[\begin{array}{rcll} \sum_{j=1}^{n} a_{ij}x_j & \leq & b_i, & \forall i \\ x_j & \geq & 0, & \forall j \end{array}\right] \iff \left[\begin{array}{rrcll} \sum_{j=1}^{n} a_{ij}x_j & +s_i & = & b_i, & \forall i \\ x_j & & \geq & 0, & \forall j \\ & s_i & \geq & 0, & \forall i \end{array}\right]$$

- The lego example:

$$\left[\begin{array}{rrcr} 2x_1 & +x_2 & \leq & 6 \\ 2x_1 & +2x_2 & \leq & 8 \\ & x_1, x_2 & \geq & 0 \end{array}\right] \iff \left[\begin{array}{rrrrcr} 2x_1 & +x_2 & \color{red}{+s_1} & & = & 6 \\ 2x_1 & +2x_2 & & \color{red}{+s_2} & = & 8 \\ & & & x_1, x_2, s_1, s_2 & \geq & 0 \end{array}\right]$$

- s_1 and s_2 are called *slack variables*—they "fill out" the (positive) distances between the left and right hand sides

The simplex method—standard form reformulations

- Surplus variables:

$$\left[\begin{array}{rcll} \displaystyle\sum_{j=1}^{n} a_{ij}x_j & \geq & b_i, & \forall i \\ x_j & \geq & 0, & \forall j \end{array}\right] \iff \left[\begin{array}{rrcll} \displaystyle\sum_{j=1}^{n} a_{ij}x_j & -s_i & = & b_i, & \forall i \\ x_j & & \geq & 0, & \forall j \\ & s_i & \geq & 0, & \forall i \end{array}\right]$$

- Surplus variable s_3 (another instance):

$$\left[\begin{array}{cccrr} x_1 & + & x_2 & \geq & 800 \\ & & x_1, x_2 & \geq & 0 \end{array}\right] \iff \left[\begin{array}{cccccrr} x_1 & + & x_2 & - & s_3 & = & 800 \\ & & & & x_1, x_2, s_3 & \geq & 0 \end{array}\right]$$

The simplex method—standard form reformulations

- Suppose that $b < 0$:

$$\begin{bmatrix} \displaystyle\sum_{j=1}^{n} a_j x_j \leq b \\ x_j \geq 0, \forall j \end{bmatrix} \Longleftrightarrow \begin{bmatrix} \displaystyle\sum_{j=1}^{n} (-a_j) x_j \geq -b \\ x_j \geq 0, \forall j \end{bmatrix} \Longleftrightarrow \begin{bmatrix} -\displaystyle\sum_{j=1}^{n} a_j x_j & -s & = -b \\ x_j & & \geq 0, \forall j \\ & s & \geq 0 \end{bmatrix}$$

- Non-negative right hand side:

$$\begin{bmatrix} x_1 - x_2 & \leq -23 \\ x_1, x_2 & \geq 0 \end{bmatrix} \Longleftrightarrow \begin{bmatrix} -x_1 + x_2 & \geq 23 \\ x_1, x_2 & \geq 0 \end{bmatrix} \Longleftrightarrow \begin{bmatrix} -x_1 + x_2 - s_4 & = 23 \\ x_1, x_2, s_4 & \geq 0 \end{bmatrix}$$

The simplex method—standard form reformulations

- Suppose that some of the variables are unconstrained (here: $k < n$). Replace x_j with $x_j^1 - x_j^2$ for the corresponding indices:

$$\left[\begin{array}{l} \displaystyle\sum_{j=1}^{n} a_j x_j \leq b \\ x_j \geq 0, j = 1, \dots, k \end{array}\right] \iff \left[\begin{array}{rl} \displaystyle\sum_{j=1}^{k} a_j x_j + \sum_{j=k+1}^{n} a_j(x_j^1 - x_j^2) + s & = b \\ x_j \geq 0, & j = 1, \dots, k, \\ x_j^1 \geq 0, x_j^2 \geq 0, & j = k+1, \dots, n \\ s \geq 0 & \end{array}\right]$$

- Sign-restricted (non-negative) variables:

$$\left[\begin{array}{rl} x_1 + x_2 & \leq 10 \\ x_1 & \geq 0 \end{array}\right] \iff \left[\begin{array}{rl} x_1 + x_2^1 - x_2^2 & \leq 10 \\ x_1, x_2^1, x_2^2 & \geq 0 \end{array}\right] \iff \left[\begin{array}{rl} x_1 + x_2^1 - x_2^2 + s_5 & = 10 \\ x_1, x_2^1, x_2^2, s_5 & \geq 0 \end{array}\right]$$

Basic feasible solutions (Ch. 4.3)

- Consider m equations with n variables, where $m \leq n$
- Set $n - m$ variables to zero and solve (if possible) the remaining $(m \times m)$ system of equations
- If the solution is *unique*, it is called a *basic* solution

Definition (Def. 4.3)

A *basic* solution to the $m \times n$ system of equations $\mathbf{Ax} = \mathbf{b}$ is obtained if $n - m$ of the variables are set to 0 and the remaining variables get their unique values from the solution to the remaining $m \times m$ system of equations.

The variables that are set to 0 are called *nonbasic variables* and the remaining m variables are called *basic variables*.

Basic feasible solutions (Ch. 4.3)

- A basic solution $\mathbf{x}$ corresponds to the *intersection* of m hyperplanes in $\mathbb{R}^m$
  - It is feasible if $\mathbf{x} \geq \mathbf{0}$
  - It is infeasible if $\mathbf{x} \not\geq \mathbf{0}$

- Each *extreme point* of the feasible set is an intersection of m hyperplanes such that all variable values are ≥ 0

- Basic feasible solution $\Longleftrightarrow$ extreme point of the feasible set

$$\begin{array}{ccc} a_{11}x_1 + \ldots + a_{1n}x_n = b_1 & \qquad & x_1 \geq 0 \\ a_{21}x_1 + \ldots + a_{2n}x_n = b_2 & & x_2 \geq 0 \\ \ldots & & \ldots \\ a_{m1}x_1 + \ldots + a_{mn}x_n = b_m & & x_n \geq 0 \end{array}$$

Basic feasible solutions

Assume that $m < n$ and that $b_i \geq 0$, $i = 1, \dots, m$, and let

$$\mathbf{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, \mathbf{A} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}, \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Consider the linear program to

$$\begin{aligned} \underset{\mathbf{x}}{\textit{minimize}} \quad & z = \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \textit{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

- Partition $\mathbf{x}$ into m basic variables $\mathbf{x}_B$ and $n - m$ non-basic variables $\mathbf{x}_N$, such that $\mathbf{x} = (\mathbf{x}_B, \mathbf{x}_N)$.
- Analogously, let $\mathbf{c} = (\mathbf{c}_B, \mathbf{c}_N)$ and $\mathbf{A} = (\mathbf{A}_B, \mathbf{A}_N) \equiv (\mathbf{B}, \mathbf{N})$
- The matrix $\mathbf{B} \in \mathbb{R}^{m \times m}$ with inverse $\mathbf{B}^{-1}$ (if it exists)

Basic feasible solutions (Ch. 4.8)

Rewrite the linear program as

$$\begin{aligned} \textit{minimize} \quad & z = \mathbf{c}_B^{\mathrm{T}}\mathbf{x}_B + \mathbf{c}_N^{\mathrm{T}}\mathbf{x}_N && \text{(1a)} \\ \textit{subject to} \quad & \mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_N = \mathbf{b} && \text{(1b)} \\ & \mathbf{x}_B \geq \mathbf{0}^m,\ \mathbf{x}_N \geq \mathbf{0}^{n-m} && \text{(1c)} \end{aligned}$$

- Multiply the system of equations (1b) by $\mathbf{B}^{-1}$ from the left:

$$\begin{aligned} \mathbf{B}^{-1}\mathbf{B}\mathbf{x}_B + \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N = \mathbf{x}_B + \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N = \mathbf{B}^{-1}\mathbf{b} \\ \Longrightarrow \mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N = \mathbf{B}^{-1}(\mathbf{b} - \mathbf{N}\mathbf{x}_N) \qquad (2) \end{aligned}$$

- Replace $\mathbf{x}_B$ in (1a) by the expression in (2):

$$\mathbf{c}_B^{\mathrm{T}}\mathbf{x}_B + \mathbf{c}_N^{\mathrm{T}}\mathbf{x}_N = \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}(\mathbf{b} - \mathbf{N}\mathbf{x}_N) + \mathbf{c}_N^{\mathrm{T}}\mathbf{x}_N = \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{b} + (\mathbf{c}_N^{\mathrm{T}} - \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{N})\mathbf{x}_N$$

$$\begin{aligned} \Rightarrow \quad \textit{minimize} \quad & z = \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{b} + (\mathbf{c}_N^{\mathrm{T}} - \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{N})\mathbf{x}_N \\ \textit{subject to} \quad & \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N \geq \mathbf{0}^m,\ \mathbf{x}_N \geq \mathbf{0}^{n-m} \end{aligned}$$

Basic feasible solutions

The rewritten program

$$\begin{aligned} \textit{minimize} \quad & z = \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{b} + (\mathbf{c}_N^{\mathrm{T}} - \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{N})\mathbf{x}_N & & & \text{(3a)} \\ \textit{subject to} \quad & \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N & \geq & \mathbf{0}^m & \text{(3b)} \\ & \mathbf{x}_N & \geq & \mathbf{0}^{n-m} & \text{(3c)} \end{aligned}$$

At the basic solution defined by $B \subset \{1, \dots, n\}$:

- Each non-basic variable takes the value 0, i.e., $\mathbf{x}_N = \mathbf{0}$
- The basic variables take the values $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N = \mathbf{B}^{-1}\mathbf{b}$
- The value of the objective function is $z = \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{b}$
- The basic solution is feasible if $\mathbf{B}^{-1}\mathbf{b} \geq \mathbf{0}^m$

The simplex method: Optimality and feasibility and change of basis (Ch. 4.4)

Optimality condition (for minimization)

The basis B is optimal if $\mathbf{c}_N^{\mathrm{T}} - \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{N} \geq \mathbf{0}^{n-m}$
(marginal values = reduced costs ≥ 0)

If not, choose as entering variable $j \in N$ the one with the lowest (negative) value of the reduced cost $c_j - \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{A}_j$

Feasibility condition

For all $i \in B$ it holds that $x_i = (\mathbf{B}^{-1}\mathbf{b})_i - (\mathbf{B}^{-1}\mathbf{A}_j)_i x_j$

Choose the leaving variable $i^* \in B$ according to

$$i^* = \arg\min_{i \in B} \left\{ \frac{(\mathbf{B}^{-1}\mathbf{b})_i}{(\mathbf{B}^{-1}\mathbf{A}_j)_i} \,\middle|\, (\mathbf{B}^{-1}\mathbf{A}_j)_i > 0 \right\}$$

Simplex search for linear optimization (Ch. 4.6)

Overview of the simplex algorithm for linear optimization (minimization)

1. **Initialization:** Choose any *feasible basis*, construct the corresponding *basic solution* $\mathbf{x}^0$, let $t = 0$
2. **Step direction:** Select a variable to *enter the basis* using the *optimality condition* (negative marginal value).
   Stop if no entering variable exists
3. **Step length:** Use the *feasibility condition* (smallest non-negative quotient) to select a variable to *leave the basis*
4. **New iterate:** Compute the *new basic solution* $\mathbf{x}^{t+1}$ by performing matrix operations
5. Let $t := t + 1$ and repeat from step 2

Basic feasible solutions, example

- Constraints:

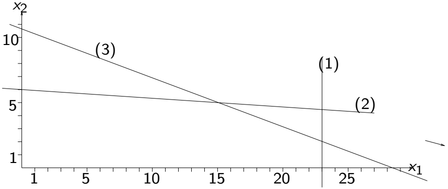
$$\begin{array}{rcrcll} x_1 & & & \leq & 23 & (1) \\ 0.067x_1 & + & x_2 & \leq & 6 & (2) \\ 3x_1 & + & 8x_2 & \leq & 85 & (3) \\ & & x_1, x_2 & \geq & 0 & \end{array}$$

- Add slack variables:

$$\begin{array}{rrrrrll} x_1 & & +s_1 & & & = 23 & (1) \\ 0.067x_1 & +x_2 & & +s_2 & & = 6 & (2) \\ 3x_1 & +8x_2 & & & +s_3 & = 85 & (3) \\ & & & & x_1, x_2, s_1, s_2, s_3 & \geq 0 & \end{array}$$

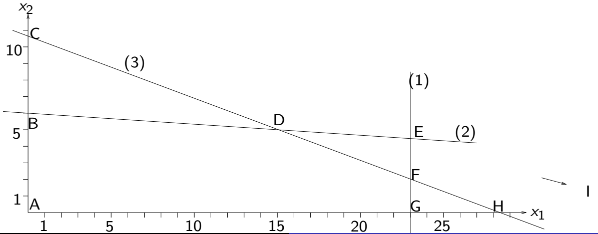
$m = 3$

$n = 5$

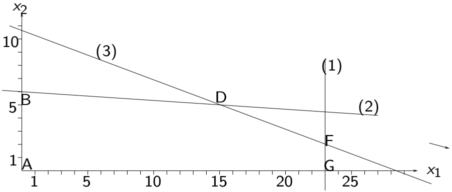


Basic and non-basic variables and solutions

basic variables	basic solution			non-basic variables $(0,0)$	point	feasible?
s_1, s_2, s_3	23	6	85	x_1, x_2	A	yes
s_1, s_2, x_1	$-5\frac{1}{3}$	$4\frac{1}{9}$	$28\frac{1}{3}$	s_3, x_2	H	no
s_1, s_2, x_2	23	$-4\frac{5}{8}$	$10\frac{5}{8}$	x_1, s_3	C	no
s_1, x_1, s_3	-67	90	-185	s_2, x_2	I	no
s_1, x_2, s_3	23	6	37	s_2, x_1	B	yes
x_1, s_2, s_3	23	$4\frac{7}{15}$	16	s_1, x_2	G	yes
x_2, s_2, s_3	-	-	-	s_1, x_1	-	-
x_1, x_2, s_1	15	5	8	s_2, s_3	D	yes
x_1, x_2, s_2	23	2	$2\frac{7}{15}$	s_1, s_3	F	yes
x_1, x_2, s_3	23	$4\frac{7}{15}$	$-19\frac{11}{15}$	s_1, s_2	E	no



Basic **feasible** solutions correspond to solutions to the system of equations that **fulfil non-negativity**



$$\begin{array}{rrrrrr} x_1 & & +s_1 & & & = 23 \\ 0.067x_1 & +x_2 & & +s_2 & & = 6 \\ 3x_1 & +8x_2 & & & +s_3 & = 85 \end{array}$$

A: $x_1 = x_2 = 0 \Rightarrow \left[\begin{array}{cccl} s_1 & & & = 23 \\ & s_2 & & = 6 \\ & & s_3 & = 85 \end{array}\right]$

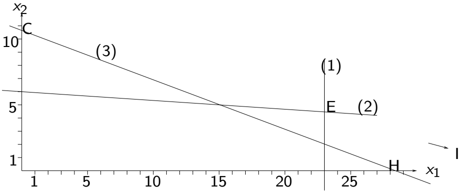
B: $x_1 = s_2 = 0 \Rightarrow \left[\begin{array}{rccl} & s_1 & & = 23 \\ x_2 & & & = 6 \\ 8x_2 & & +s_3 & = 85 \end{array}\right]$

D: $s_3 = s_2 = 0 \Rightarrow \left[\begin{array}{rrrl} x_1 & & +s_1 & = 23 \\ 0.067x_1 & +x_2 & & = 6 \\ 3x_1 & +8x_2 & & = 85 \end{array}\right]$

F: $s_3 = s_1 = 0 \Rightarrow \left[\begin{array}{rrrl} x_1 & & & = 23 \\ 0.067x_1 & +x_2 & +s_2 & = 6 \\ 3x_1 & +8x_2 & & = 85 \end{array}\right]$

G: $x_2 = s_1 = 0 \Rightarrow \left[\begin{array}{rrrl} x_1 & & & = 23 \\ 0.067x_1 & +s_2 & & = 6 \\ 3x_1 & & +s_3 & = 85 \end{array}\right]$

Basic **infeasible** solutions corresp. to solutions to the system of equations with one or more variables < 0



$$\begin{array}{rrrrrl} x_1 & & +s_1 & & & = 23 \\ 0.067x_1 & +x_2 & & +s_2 & & = 6 \\ 3x_1 & +8x_2 & & & +s_3 & = 85 \end{array}$$

$$\text{H: } x_2 = s_3 = 0 \Rightarrow \left[\begin{array}{rrrl} x_1 & +s_1 & & = 23 \\ 0.067x_1 & & +s_2 & = 6 \\ 3x_1 & & & = 85 \end{array}\right]$$

$$\text{C: } x_1 = s_3 = 0 \Rightarrow \left[\begin{array}{rrrl} & s_1 & & = 23 \\ x_2 & & +s_2 & = 6 \\ 8x_2 & & & = 85 \end{array}\right]$$

$$\text{I: } s_2 = x_2 = 0 \Rightarrow \left[\begin{array}{rrrl} x_1 & +s_1 & & = 23 \\ 0.067x_1 & & & = 6 \\ 3x_1 & & +s_3 & = 85 \end{array}\right]$$

$$\text{-: } s_1 = x_1 = 0 \Rightarrow \left[\begin{array}{rrrl} & & 0 & = 23 \\ x_2 & +s_2 & & = 6 \\ 8x_2 & & +s_3 & = 85 \end{array}\right]$$

$$\text{E: } s_1 = s_2 = 0 \Rightarrow \left[\begin{array}{rrrl} x_1 & & & = 23 \\ 0.067x_1 & +x_2 & & = 6 \\ 3x_1 & +8x_2 & +s_3 & = 85 \end{array}\right]$$

Basic feasible solutions and the simplex method

- Express the m *basic* variables in terms of the $n - m$ *non-basic* variables

Example: Start at $x_1 = x_2 = 0 \Rightarrow s_1, s_2, s_3$ are *basic*

$$\begin{array}{rrrrrl} x_1 & & +s_1 & & & = 23 \\ \frac{1}{15}x_1 & +x_2 & & +s_2 & & = 6 \\ 3x_1 & +8x_2 & & & +s_3 & = 85 \end{array}$$

Express s_1, s_2, and s_3 in terms of x_1 and x_2 (*non-basic*):

$$\begin{array}{rcrrr} s_1 & = & 23 & -x_1 & \\ s_2 & = & 6 & -\frac{1}{15}x_1 & -x_2 \\ s_3 & = & 85 & -3x_1 & -8x_2 \end{array}$$

- We wish to maximize the value of the objective function $2x_1 + 3x_2$

Express the objective in terms of the *non-basic* variables:

$$\text{(maximize)} \quad z = 2x_1 + 3x_2 \quad \Leftrightarrow \quad z - 2x_1 - 3x_2 = 0$$

Basic feasible solutions and the simplex method

The *first basic solution* can be represented as

$$\begin{array}{rrrrrrl|l|}
-z & +2x_1 & +3x_2 & & & & = 0 & (0) \\
 & x_1 & & +s_1 & & & = 23 & (1) \\
 & \frac{1}{15}x_1 & +x_2 & & +s_2 & & = 6 & (2) \\
 & 3x_1 & +8x_2 & & & +s_3 & = 85 & (3)
\end{array}$$

- Marginal values for increasing the non-basic variables x_1 and x_2 from zero: 2 and 3, resp.

$\Rightarrow$ Choose x_2 — let x_2 *enter the basis* DRAW GRAPH!!

- One basic variable (s_1, s_2, or s_3) must *leave the basis*. Which?

The value of x_2 increases until a basic variable reaches the value 0:

$$\left.\begin{array}{ll}
(2): s_2 = 6 - x_2 \geq 0 & \Rightarrow x_2 \leq 6 \\
(3): s_3 = 85 - 8x_2 \geq 0 & \Rightarrow x_2 \leq 10\frac{5}{8}
\end{array}\right\} \Rightarrow \begin{array}{c} s_2 = 0 \text{ when } x_2 = 6 \\ (\text{and } s_3 = 37) \end{array}$$

- s_2 will leave the basis

Change basis through row operations

Eliminate s_2 from the basis let x_2 enter the basis—use row operations:

$$
\begin{array}{rrrrrrcr|l}
-z & +2x_1 & +3x_2 & & & & = & 0 & (0) \\
 & x_1 & & +s_1 & & & = & 23 & (1) \\
 & \frac{1}{15}x_1 & +x_2 & & +s_2 & & = & 6 & (2) \\
 & 3x_1 & +8x_2 & & & +s_3 & = & 85 & (3) \\
\hline
-z & +\frac{9}{5}x_1 & & & -3s_2 & & = & -18 & (0)-3\cdot(2) \\
 & x_1 & & +s_1 & & & = & 23 & (1)-0\cdot(2) \\
 & \frac{1}{15}x_1 & +x_2 & & +s_2 & & = & 6 & (2) \\
 & \frac{37}{15}x_1 & & & -8s_2 & +s_3 & = & 37 & (3)-8\cdot(2)
\end{array}
$$

- Corresponding basic solution: $s_1 = 23$, $x_2 = 6$, $s_3 = 37$.
- Nonbasic variables: $x_1 = s_2 = 0$
- The marginal value of x_1 is $\frac{9}{5} > 0$. Let x_1 enter the basis
- Which one should leave? s_1, x_2, or s_3?

Change basis ... x_1 enters the basis (marginal value > 0)

$$\begin{array}{rrrrrrcr|l}
-z & +\frac{9}{5}x_1 & & & -3s_2 & & = & -18 & (0)\\
 & x_1 & & +s_1 & & & = & 23 & (1)\\
 & \frac{1}{15}x_1 & +x_2 & & +s_2 & & = & 6 & (2)\\
 & \boldsymbol{\frac{37}{15}x_1} & & & -8s_2 & +s_3 & = & 37 & (3)
\end{array}$$

The value of x_1 increases until a basic variable reaches the value 0:

$$\left.\begin{array}{ll}
(1): s_1 = 23 - x_1 \geq 0 & \Rightarrow x_1 \leq 23\\
(2): x_2 = 6 - \frac{1}{15}x_1 \geq 0 & \Rightarrow x_1 \leq 90\\
(3): s_3 = 37 - \frac{37}{15}x_1 \geq 0 & \Rightarrow x_1 \leq 15
\end{array}\right\} \Rightarrow \begin{array}{c} s_3 = 0 \text{ when}\\ x_1 = 15 \end{array}$$

x_1 enters and s_3 leaves the basis: perform row operations:

$$\begin{array}{rrrrrrcr|l}
-z & & & & +2.84s_2 & -0.73s_3 & = & -45 & (0)-(3)\cdot\frac{15}{37}\cdot\frac{9}{5}\\
 & & & s_1 & +3.24s_2 & -0.41s_3 & = & 8 & (1)-(3)\cdot\frac{15}{37}\\
 & & x_2 & & +1.22s_2 & -0.03s_3 & = & 5 & (2)-(3)\cdot\frac{15}{37}\cdot\frac{1}{15}\\
 & x_1 & & & -3.24s_2 & +0.41s_3 & = & 15 & (3)\cdot\frac{15}{37}
\end{array}$$

Change basis ... s_2 enters the basis (marginal value > 0)

$$
\begin{array}{llllrrcr|l}
-z & & & & +2.84s_2 & -0.73s_3 & = & -45 & (0) \\
 & & & s_1 & +\mathbf{3.24s_2} & -0.41s_3 & = & 8 & (1) \\
 & & x_2 & & +1.22s_2 & -0.03s_3 & = & 5 & (2) \\
 & x_1 & & & -3.24s_2 & +0.41s_3 & = & 15 & (3)
\end{array}
$$

The value of s_2 increases until some basic variable value $= 0$:

$$
\left.
\begin{array}{ll}
(1): s_1 = 8 - 3.24s_2 \geq 0 & \Rightarrow s_2 \leq 2.47 \\
(2): x_2 = 5 - 1.22s_2 \geq 0 & \Rightarrow s_2 \leq 4.10 \\
(3): x_1 = 15 + 3.24s_2 \geq 0 & \Rightarrow s_2 \geq -4.63
\end{array}
\right\}
\Rightarrow
\begin{array}{c}
s_1 = 0 \text{ when} \\
s_2 = 2.47
\end{array}
$$

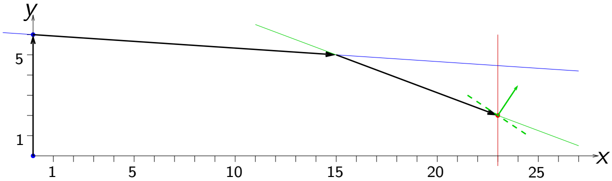
s_2 enters and s_1 leaves the basis: perform row operations

$$
\begin{array}{lllrlr|cr|l}
-z & & & -0.87s_1 & & -0.37s_3 & = & -52 & (0)-(1)\cdot\frac{2.84}{3.24} \\
 & & & 0.31s_1 & +s_2 & -0.12s_3 & = & 2.47 & (1)\cdot\frac{1}{3.24} \\
 & & x_2 & -0.37s_1 & & +0.12s_3 & = & 2 & (2)-(1)\cdot\frac{1.22}{3.24} \\
 & x_1 & & +s_1 & & & = & 23 & (3)+(1)
\end{array}
$$

Optimal basic solution

$$\begin{array}{rrrrrrcr}
-z & & & -0.87s_1 & & -0.37s_3 & = & -52 \\
 & & & 0.31s_1 & +s_2 & -0.12s_3 & = & 2.47 \\
 & & x_2 & -0.37s_1 & & +0.12s_3 & = & 2 \\
 & x_1 & & +s_1 & & & = & 23
\end{array}$$

- No marginal value is positive. No improvement can be made
- The optimal basis is given by $s_2 = 2.47$, $x_2 = 2$, and $x_1 = 23$
- Non-basic variables: $s_1 = s_3 = 0$
- Optimal value: $z = 52$



Summary of the solution course

basis	$-z$	x_1	x_2	s_1	s_2	s_3	RHS
$-z$	1	2	3	0	0	0	0
s_1	0	1	0	1	0	0	23
s_2	0	0.067	**1**	0	1	0	6
s_3	0	3	8	0	0	1	85
$-z$	1	1.80	0	0	-3	0	-18
s_1	0	1	0	1	0	0	23
x_2	0	0.07	1	0	1	0	6
s_3	0	**2.47**	0	0	-8	1	37
$-z$	1	0	0	0	2.84	-0.73	-45
s_1	0	0	0	1	**3.24**	-0.41	8
x_2	0	0	1	0	1.22	-0.03	5
x_1	0	1	0	0	-3.24	0.41	15
$-z$	1	0	0	-0.87	0	-0.37	-52
s_2	0	0	0	0.31	1	-0.12	2.47
x_2	0	0	1	-0.37	0	0.12	2
x_1	0	1	0	1	0	0	23

Homework

Solve the lego problem using the simplex method

$$\begin{array}{rrcrcrcr}
\text{maximize} & z & = & 1600x_1 & + & 1000x_2 & & \\
\text{subject to} & & & 2x_1 & + & x_2 & \leq & 6 \\
 & & & 2x_1 & + & 2x_2 & \leq & 8 \\
 & & & & & x_1,\ x_2 & \geq & 0
\end{array}$$