MVE165/MMG631
Linear and Integer Optimization with Applications
Lecture 4
Linear programming: degeneracy; unbounded solution; infeasibility; starting solutions

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Properties of linear optimization problems that are utilized for the Simplex method

- **Optimality condition**: The *entering* variable in a minimization (maximization) problem should have the largest negative (positive) marginal value (reduced cost).

  The entering variable *determines a direction* in which the objective value increases (decreases) the fastest.

  This direction is *along an edge* of the feasible polyhedron.

  If all *reduced costs are positive* (negative), then the current basis is *optimal*.

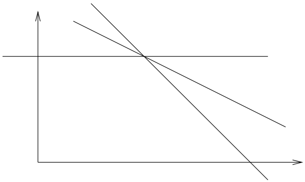
- **Feasibility condition**: The *leaving* variable is the one with smallest nonnegative quotient.

  Corresponds to the constraint that would be *violated first*

Degeneracy (Ch. 4.10)

- If the smallest nonnegative quotient is zero, the value of a basic variable will become zero in the next iteration
- The solution is *degenerate*
- The objective value will *not* improve in this iteration
- Risk: *cycling* around (non-optimal) bases
- Reason: a *redundant* constraint "touches" the feasible set
- Example:

$$\begin{array}{rcrcl} x_1 & + & x_2 & \leq & 6 \\ & & x_2 & \leq & 3 \\ x_1 & + & 2x_2 & \leq & 9 \\ & & x_1,\ x_2 & \geq & 0 \end{array}$$



Convergence of the simplex algorithm (Ch. 4.10)

Finite convergence of the simplex algorithm

If all of the basic feasible solutions are non-degenerate, then the simplex algorithm terminates after a finite number of iterations.

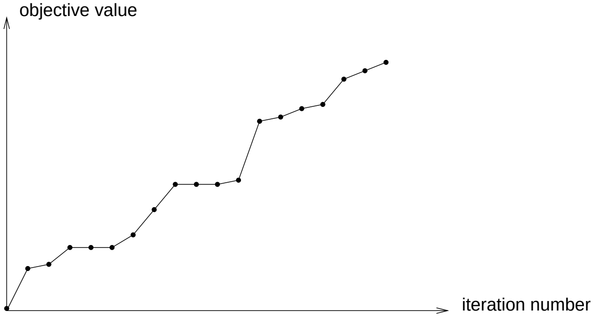
Proof (rough argument)

Non-degeneracy implies that the step length is > 0; hence, we cannot return to an old basic feasible solution once we have left it. There are finitely many basic feasible solutions

- Degeneracy can actually lead to cycling—the same sequence of basic feasible solutions is repeated infinitely.
- Remedy: Change the incoming/outgoing criteria! Bland's rule: Sort variables according to some index ordering.

Degeneracy

- Typical objective function progress (maximization) of the simplex method



- In modern software: perturb the right hand side $(b_i + \Delta b_i)$ – solve – reduce the perturbation – resolve starting from the current basis – repeat until $\Delta b_i = 0$

Multiple optimal solutions

- If the entering variable has a *zero reduced cost*, then there are (at least) two optimal extreme points
- Also all points on the edge between two optimal extreme points are optimal
  - How does this generalize when there are three or more optimal extreme points?

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Unbounded solutions (Ch. 4.4, 4.6)

- If all quotients are *negative*, the value of the variable entering the basis may increase *infinitely*
- Then, the feasible set is *unbounded*
- In a real application this would probably be due to some incorrect assumption (recall "the process of optimization").
- Example:

$$
\begin{aligned}
\text{minimize} \quad z = \quad & -x_1 && -2x_2 && && \text{(1a)}\\
\text{subject to} \quad & -x_1 && +x_2 && \leq 2 && \text{(1b)}\\
& -2x_1 && +x_2 && \leq 1 && \text{(1c)}\\
& && x_1,\ x_2 && \geq 0 && \text{(1d)}
\end{aligned}
$$

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Unbounded solutions (Ch. 4.4, 4.6)

- A feasible basis of the problem (1) is given by $x_1 = 1$, $x_2 = 3$, with corresponding tableau[1]

basis	$-z$	x_1	x_2	s_1	s_2	RHS
$-z$	1	0	0	5	-3	7
x_1	0	1	0	1	-1	1
x_2	0	0	1	2	-1	3

- Entering variable is s_2
- Row 1: $x_1 = 1 + s_2 \geq 0 \quad \Longrightarrow \quad s_2 \geq -1$
- Row 2: $x_2 = 3 + s_2 \geq 0 \quad \Longrightarrow \quad s_2 \geq -3$
- No leaving variable can be found, since no constraint will prevent s_2 from increasing infinitely
- The problem has an *unbounded* solution

[1] *Homework: Find this basis using the simplex method*

Find an initial basic feasible solution—phase I

- If an initial basic feasible solution cannot be easily found:
- Assume that $\mathbf{b} \geq \mathbf{0}^m$. Introduce an *artificial variable* a_i in each row that lacks a unit column
- Solve the *phase I-problem*:

$$
\begin{array}{lrl}
\text{minimize} & w = & (\mathbf{1}^m)^{\mathrm{T}}\mathbf{a} \\
\text{subject to} & \mathbf{A}\mathbf{x} + \mathbf{I}^m\mathbf{a} & = \mathbf{b}, \\
& \mathbf{x} & \geq \mathbf{0}^n, \\
& \mathbf{a} & \geq \mathbf{0}^m
\end{array}
$$

Find an initial basic feasible solution—phase II

- The case when feasible solutions exist
  - $w^* = 0$, meaning that $\mathbf{a}^* = \mathbf{0}^m$ must hold
  - The resulting basic feasible solution is feasible in the *original* problem and optimal in the phase I-problem
  - Start phase-II: solve the original problem, starting from this basic feasible solution

- The case when feasible solutions do not exist
  - $w^* > 0$. The optimal basis then has some $a_i^* > 0$
  - Due to the construction of the objective function, there exists no basic feasible solution in the original problem
  - The originalproblem is then infeasible
  - What to do then? Modelling errors? Can be detected from the optimal solution. In fact, some linear optimization problems are pure feasibility problems.